Boundary conditions and subelliptic estimates for geometric Kramers-Fokker-Planck operators on manifolds with boundaries

Francis Nier

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Abstract

This article is concerned with the maximal accretive realizations of geometric Kramers-Fokker-Planck operators on manifolds with boundaries. A general class of boundary conditions is introduced which ensures the maximal accretivity and some global subelliptic estimates. Those estimates imply nice spectral properties as well as exponential decay properties for the associated semigroup. Admissible boundary conditions cover a wide range of applications for the usual scalar Kramer-Fokker-Planck equation or Bismut's hypoelliptic laplacian.

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1 Introduction

1.1 Motivations

A few years ago, while I was visiting the WIAS in Berlin, Holger Stephan asked me about the relevant boundary conditions for the Kramers-Fokker-Planck operator

$$p.\partial_q - (\partial_q V(q)).\partial_p + \frac{-\Delta_p + |p|^2}{2}$$

in $\Omega \times \mathbb{R}^d_p$ when $\Omega \subset \mathbb{R}^d_q$ is a bounded regular domain. At that time I was aware of the previous works developed within the mathematical analysis of kinetic models (see a.e. [DeMG][Car][Luc]) but I noticed, and I have kept in mind since, that the weak formulation (of time-dependent problems) were far from providing the accurate semigroup or resolvent information which was accessible in the boundaryless case (see e.g. [HerNi][HelNi]). After this various things occurred:

- We realized recently with D. Le Peutrec and C. Viterbo in [LNV] that the introduction of artificial boundary value problems was an important step in the accurate spectral analysis of Witten Laplacians acting on p-forms in the low temperature limit. Actually the harmonic forms of these artificial boundary value problems, after Witten's deformation, play in the case of p-forms, $p > 1$, the role of the truncated version $\chi(q)e^{-V(q)/h}$ of the explicit equilibrium density $e^{-V(q)/h}$ used in [HKN] for the case $p = 0$. The Kramers-Fokker-Planck operator is actually the $p = 0$ version of Bismut's hypoelliptic Laplacian. We may hope to extend the accurate spectral analysis of the case $p = 0$ done by Hérau-Hitrik-Sjöstrand in [HHS2], if we understand the suitable realizations of the hypoelliptic Laplacian on a manifold with boundary.

- While we were working with T. Lelièvre about quasi-stationnary distributions for the Einstein-Smoluchowski case (SDE in a gradient field), with the help of boundary Witten Laplacians acting on functions ($p = 0$) and 1-forms, T. Lelièvre repeatedly insisted that the Langevin process was a more natural process for molecular dynamics. The quasi-stationnary distributions are especially used to develop or study efficients algorithms for molecular dynamics. The PDE formulation of the Langevin process is the Kramers-Fokker-Planck equation and quasi-stationnary distributions especially make sense on bounded domains (in the position variable q).

- A bit more than one year ago, we had discussions with F. Hérau and D. Le Peutrec who elaborated an approach of those boundary value problems relying on the weak formulation of [Car] combined with the so-called hypocoercive techniques proposed by C. Villani in [Vil]. They were embarrassed with the definition of traces and I suggested that the problem should be reconsidered from the beginning, by trying to mimic what is done for elliptic operators with the introduction of Calderon projectors. We continued on our different ways while keeping in touch. Their point of view may be efficient for nonlinear problems.

- In 2007, G. Lebeau besides his work [BiLe] in collaboration with J.M. Bismut about the hypoelliptic Laplacian, proved maximal estimates for what he called the geometric (Kramers)-Fokker-Planck operator. This is actually the scalar principal part of the hypoelliptic Laplacian. Those estimates have not been, seemingly, employed seriously up to now. We shall see that they are instrumental in absorbing some singular perturbations due to the curvature of the boundary.

This article presents the functional analysis of geometric Kramers-Fokker-Planck operators with a rather wide and natural class of boundary conditions. About the application to Bismut's hypoelliptic Laplacian and if one compares with the program which has been achieved for the accurate asymptotic and spectral analysis of boundary Witten Laplacians in [HeNi1] and [Lep3] and its applications in [LNV] and [LeNi], it is only the beginning. Nothing is said about the asymptotic analysis with respect to small parameters, nor about the supersymmetric arguments so effective when dealing with the Witten Laplacian. In the case of the Kramers-Fokker-Planck equation two parameters can be introduced with independent or correlated asymptotics, the temperature and the friction (see for example [Ris][HerNi]).
Other boundary conditions which are proposed as examples of application,

are heuristically introduced by completing the Langevin process with a jump process when particles hit the boundary. Except in the case of specular reflection studied in [Lap][BoJa] or exotic one-dimensional problems with non elastic boundary conditions in [Ber], no definite result seems available for justifying these heuristic arguments. Our subelliptic (regularity and decay) estimates of the corresponding boundary value problem, may help to get a better understanding.
Finally one may wonder whether it is necessary to work in the general framework of riemannian geometry. Our approach passes through the local reduction to straight half-space problem where the coordinate changes prevent us from sticking to the euclidean case. Moreover, even when the metric is flat (the Riemann curvature tensor vanishes), the extrinsic curvature of the boundary raises crucial difficulties in the analysis. Working with a general metric on a riemannian manifold with boundary (in the q variable) does not add any serious difficulties. I nevertheless chose to put the stress on a presentation in coordinate systems, in order to make the essential points of the analysis more obvious, and to avoid confusing the possibly non familiar reader with the concise but nevertheless subtle notations of intrinsic geometry. Such formulations occur when necessary in the end of the article, for example while applying the general framework to Bismut's hypoelliptic Laplacian.

1.2 The problem

We shall consider the geometric Kramers-Fokker-Planck equation set on $X = T^*Q$ when $\overline{Q} = Q \cup \partial Q$ is a d-dimensional compact riemannian manifold with boundary or a compact perturbation of the euclidean half-space $\overline{\mathbb{R}^{2d}_-} = (-\infty, 0] \times \mathbb{R}^{d-1} \times \mathbb{R}^d$. When $q \in \partial Q$, the fiber $T^*_q Q$ is the direct sum $T^*_q \partial Q \oplus N^*_{q,\partial Q} Q$ where $N^*_{q,\partial Q} Q$ is the conormal fiber at q.
In a neighborhood U of $q \in \overline{Q}$, position coordinates are denoted by $(q^1, \ldots, q^d)$.
We shall use Einstein's convention of up and down repeated indices. An element of the fiber $p \in T^*_q Q$, $q \in U$, is written $p = p_i dq^i$ and $(q^1, \ldots, q^d, p_1, \ldots, p_d)$ are symplectic coordinates in $U \times \mathbb{R}^d \sim T^*U \subset T^*Q$.
The metric on $\overline{Q}$, i.e. on the tangent fiber bundle $\pi_{TQ} : TQ \to Q$, is denoted by $g(q) = g^T(q) = (g_{ij}(q))_{1 \leq i,j \leq d}$ or $g = g_{ij}(q) dq^i dq^j$, and its dual metric on the cotangent bundle $\pi_{T^*Q} : T^*Q \to Q$ is $g^{-1}(q) = (g^{ij}(q))_{1 \leq i,j \leq d}$.
The cotangent bundle $X = T^*Q$ or $\overline{X} = T^*\overline{Q} = X \sqcup \partial X$, viewed as a man-

ifold, is endowed with the metric $g \oplus g^{-1}$. Actually for every $x \in \overline{X}$, the tangent space $T_xX = T_xT^*Q$ is decomposed into an horizontal and vertical component $T_xT^*Q = (T_xT^*Q)^H \oplus (T_xT^*Q)^V$ (see Section 6 for details) which specifies the orthogonal decomposition of the metric $g \oplus g^{-1}$. The corresponding volume on $X = T^*Q$ coincides with the symplectic volume form $\frac{(-1)^d}{d!}(dp_i \wedge dq^i)^{\wedge d}$ on T^*Q, and the integration measure is, locally in T^*U with symplectic coordinates, the $\mathbb{R}^{2d}$-Lebesgue's measure simply denoted by $dqdp$. The L^2-space on X will be denoted $L^2(X, dqdp)$ and the scalar product (extended as a duality product between distributions and test functions) and the norm are

$$\langle u\,,\, v\rangle = \int_X \overline{u(q,p)}v(q,p)\; dqdp \quad , \quad \|u\|^2 = \langle u\,,\, v\rangle\,.$$

We shall consider also $\mathfrak{f}-$valued functions on $\overline{X}$, where $\mathfrak{f}$ is a complex Hilbert-space[1]. The above scalar (duality) product has to be replaced by

$$\langle u\,,\, v\rangle = \int_X \langle u(q,p)\,,\, v(q,p)\rangle_{\mathfrak{f}}\; dqdp\,,$$

and the corresponding L^2-space will be denoted $L^2(X, dqdp; \mathfrak{f}) = L^2(X, dqdp) \otimes \mathfrak{f}$ while the space of compactly supported regular sections on $\overline{X}$ (resp. X) is denoted by $\mathcal{C}_0^\infty(\overline{X}; \mathfrak{f})$ (resp. $\mathcal{C}_0^\infty(X; \mathfrak{f})$). Hilbert-completed tensor products of two Hilbert spaces $\mathfrak{f}_1\,,\mathfrak{f}_2$, will be denoted by $\mathfrak{f}_1 \otimes \mathfrak{f}_2$. When necessary the algebraic tensor product (or other completions) will be specified by a notation like $\mathfrak{f}_1 \otimes^{alg} \mathfrak{f}_2$.
In a vertical fiber $T_q^*\overline{Q}$ endowed with the scalar product $g^{-1}(q)$, the length $|p|_q$, the vertical Laplacian Δ_p and the harmonic oscillator hamiltonian are defined by

$$|p|_q^2 = g^{ij}(q)p_ip_j = p^T g^{-1}(q)p\,,$$
$$\Delta_p = \partial_{p_i} g_{ij}(q)\partial_{p_j} \quad , \quad \mathcal{O}_g = \frac{-\Delta_p + |p|_q^2}{2}\,.$$

With the energy $\mathcal{E}(q,p) = \frac{1}{2}|p|_q^2$ defined on T^*Q, we associate the Hamiltonian vector field

$$\mathcal{Y}_{\mathcal{E}} = g^{ij}(q)p_i\partial_{q^j} - \frac{1}{2}\partial_{q^k} g^{ij}(q)p_ip_j\partial_{p_k}\,,$$

[1]All our Hilbert spaces are assumed separable

The geometric Kramers-Fokker-Planck operator that we consider is the scalar (even when $\mathfrak{f} \neq \mathbb{C}$) differential operator

$$P_{\pm,Q,g} = \pm \mathcal{Y}_{\mathcal{E}} + \mathcal{O}_g \,,$$

well-defined on $\mathcal{C}_0^\infty(\overline{X};\mathfrak{f})$. Natural functional spaces are associated with the vertical operator $\mathcal{O}_g$. Along a vertical fiber they are the Sobolev spaces associated with the harmonic oscillator hamiltonian $\mathcal{O}_g(q) = \frac{-\Delta_p + |p|_q^2}{2}$:

$$\mathcal{H}^s(q) = \left\{ u \in \mathcal{S}'(\mathbb{R}^d, dp; \mathfrak{f}) \,, \quad (\frac{d}{2} + \mathcal{O}_g(q))^{s/2} u \in L^2(\mathbb{R}^d, dp; \mathfrak{f}) \right\} . \quad (1)$$

The distributional space $\mathcal{H}^s(q)$ coincides with the same space given by the euclidean distance on $\mathbb{R}^d_p$, owing to the global ellipticity of $(\frac{d}{2} + \mathcal{O}_{g(q)})$ (see [Hel1][HormIII]-Chap 18). Only the corresponding norm depends on $g(q)$, via a simple linear change of the variable p, and this provides a Hermitian bundle structure on $\pi_{\mathcal{H}^s} : \mathcal{H}^s \to \overline{Q}$, $\pi_{\mathcal{H}^s}^{-1}(q) = \mathcal{H}^s(q)$. Global norms on $L^2(Q;\mathcal{H}^1)$ and $L^2(Q;\mathcal{H}^2)$ are given by

$$\|u\|^2_{L^2(Q;\mathcal{H}^1)} = \frac{1}{2}\int_X \|g^{1/2}(q)\partial_p u(q,p)\|^2_{\mathfrak{f}} + \|g(q)^{-1/2} p u(q,p)\|^2_{\mathfrak{f}} \; dqdp + d\|u\|^2$$

$$\|u\|^2_{L^2(Q;\mathcal{H}^2)} = \|(\frac{d}{2} + \mathcal{O}_g)u\|^2 \,.$$

The standard Sobolev spaces on $\overline{Q}$ are denoted by $H^s(\overline{Q})$, $s \in \mathbb{R}$ and the corresponding spaces of $\mathcal{H}^{s'}$-sections are denoted by $H^s(\overline{Q};\mathcal{H}^{s'})$, $s, s' \in \mathbb{R}$. According to [ChPi], the spaces $H^s(\overline{Q})$ are locally identified with $H^s(\overline{\mathbb{R}^d_-}) = H^s((-\infty, 0] \times \mathbb{R}^{d-1})$, which is the set of $u \in \mathcal{D}'(\mathbb{R}^d_-) = \mathcal{D}'((-\infty,0) \times \mathbb{R}^{d-1})$ such that $u = \tilde{u}\big|_{\mathbb{R}^d_-}$ with $\tilde{u} \in H^s(\mathbb{R}^d)$. The definitions of $\mathcal{C}_0^\infty(\overline{Q})$ and $\mathcal{C}_0^\infty(\overline{X};\mathfrak{f})$ follow the same rule. Similarly we shall use the notations $\mathcal{S}(\overline{\mathbb{R}^d_-})$, $\mathcal{S}(\overline{\mathbb{R}^{2d}_-})$.
In order to study closed (and maximal accretive realizations) of $P_{\pm,Q,g}$ we need to specify boundary conditions. In a neighborhood $U \subset \overline{Q}$ of $q \in \partial Q$, coordinates can be chosen so that the metric g equals

$$g(q) = \begin{pmatrix} 1 & 0 \\ 0 & m_{ij}(q^1, q') \end{pmatrix} \quad , \quad g_{ij}(q)dq^i dq^j = (dq^1)^2 + m_{ij}(q^1,q')dq'^i dq'^j \,. \quad (2)$$

The corresponding coordinates on T^*U are (q^1, q', p_1, p') and (q', p_1) is a coordinate system for the conormal bundle $N^*_{\partial Q}\overline{Q}$. The symplectic volume element is

$$dqdp = dq_1 dp_1 dq' dp' \,.$$

On $\partial X = T^*Q\big|_{\partial Q}$ we shall use the measure $|p_1|dp_1 dq' dp'$ and the corresponding L^2-space will be denoted $L^2(\partial X, |p_1|dq'dp; \mathfrak{f})$ with the norm locally defined by

$$\|\gamma\|^2_{L^2(T^*_{Q'}U, |p_1|dq'dp;\mathfrak{f})} = \int_{T^*_{Q'}U} \|\gamma(q,p)\|^2_{\mathfrak{f}} \; |p_1|dq'dp \,,$$

This definition does not depend on the choice of coordinates $(q^1, \ldots, q^d)$ for the decomposition (2) summarized as $g = 1 \oplus^{\perp} m$. Polar coordinates in the momentum variable can be defined globally as well. By writing the momentum $p \in T^*_q Q$, $p = r\omega$ with $r = |p|_q$ and $\omega \in S^*_q Q$, the space $L^2(\partial X, |p_1|dq'dp; \mathfrak{f})$ equals

$$\begin{aligned} L^2(T^*_{\partial Q}Q, |p_1|dq'dp; \mathfrak{f}) &= L^2((0,+\infty), r^{d-1}dr; L^2(S^*_{\partial Q}Q, |\omega_1|dq'd\omega; \mathfrak{f})) \\ &= L^2(\partial Q \times (0,+\infty), r^{d-1}dr dq'; L^2(S^*_q Q, |\omega_1|d\omega; \mathfrak{f}))\,. \end{aligned}$$

Once the decomposition $g = 1 \oplus^{\perp} m$ is assumed, the last line defines a space of L^2-sections of a Hilbert bundle on $\partial Q \times (0, +\infty)$.
On those coordinates, the mappings

$$\begin{array}{rl} & (q', p_1) \to (q', -p_1) \\ \text{(resp.)} & (q', p_1, p') \to (q', -p_1, p') \end{array}$$

defines an involution on the conormal bundle $N^*_{\partial Q}Q$ (resp. on $T^*_{\partial Q}Q$). On $\partial X = T^*_{\partial Q}Q$ this involution preserves the energy shells $\left\{|p|^2_q = C\right\}$. The function $\operatorname{sign}(p_1)$ is well-defined on $N^*_{\partial Q}Q$ and $T^*_{\partial Q}Q$, $p_1 > 0$ corresponding to the exterior conormal orientation. When the space $\mathfrak{f}$ is endowed with a unitary involution j, the mapping

$$v(q', p_1, p') \to jv(q', -p_1, p')$$

defines a unitary involution on $L^2(\partial X, |p_1|dq'dp; \mathfrak{f})$. The even and odd part of $\gamma \in L^2(\partial X, |p_1|dq'dp; \mathfrak{f})$ are given by

$$\gamma_{ev}(q', p_1, p') = [\Pi_{ev}\gamma](q', p) = \frac{\gamma(q', p_1, p') + j\gamma(q', -p_1, p')}{2}\,, \tag{3}$$

$$\gamma_{odd}(q', p_1, p') = [\Pi_{odd}\gamma](q', p) = \frac{\gamma(q', p_1, p') - j\gamma(q', -p_1, p')}{2}\,. \tag{4}$$

The operator Π_{ev} and $\Pi_{odd}=1-\Pi_{ev}$ are pointwise operations in $(q',|p|_q)$ which can be written as operator valued multiplications by $\Pi_{odd,ev}(q',|p|_q)$ and they are orthogonal projections in

$$L^2(\partial X,|p_1|dq'dp;\mathfrak{f})=L^2(\partial Q\times(0,+\infty),r^{d-1}drdq';L^2(S^*_qQ,|\omega_1|d\omega;\mathfrak{f}))\,.$$

We shall also use a bounded accretive operator A on $L^2(\partial X,|p_1|dq'dp;\mathfrak{f})$ which is also diagonal in the variable $(q',|p|_q)$. We shall assume that there exists $\|A\|>0$ and $c_A>0$ such that

$$\begin{aligned}
&\|A(q,r)\|_{\mathcal{L}(L^2(S^*_qQ,|\omega_1|d\omega;\mathfrak{f}))}\leq\|A\|\quad\text{for a.e. }(q,r)\in\partial Q\times\mathbb{R}_+\,, && (5)\\
&[A(q,r)\,,\,\Pi_{ev}(q,r)]=0\quad\text{for a.e. }(q,r)\in\partial Q\times\mathbb{R}_+\,, && (6)\\
\text{with either}\quad&\min\sigma(\mathbb{R}\mathrm{e}\;A(q,r))\geq c_A>0\quad\text{for a.e. }(q,r)\in\partial Q\times\mathbb{R}_+\,, && (7)\\
\text{or}\quad&A(q,r)=0\quad\text{for a.e. }(q,r)\in\partial Q\times\mathbb{R}_+\,. && (8)
\end{aligned}$$

Our aim is to study the operator $K_{\pm,A,g}$ defined in $L^2(X,dqdp;\mathfrak{f})$ with the domain $D(K_{\pm,A,g})$ characterized by

$$\begin{aligned}
&u\in L^2(Q,dq;\mathcal{H}^1)\quad,\quad P_{\pm,Q,g}u\in L^2(X,dqdp;\mathfrak{f})\,, && (9)\\
&\forall R>0\,,1_{[0,R]}(|p|_q)\gamma u\in L^2(\partial X,|p_1|dq'dp;\mathfrak{f})\,, && (10)\\
&\gamma_{odd}u=\pm\,\mathrm{sign}\,(p_1)A\gamma_{ev}u\,. && (11)
\end{aligned}$$

For the case $A=0$, it is interesting to introduce the set $\mathcal{D}(\overline{X},j)$ of regular functions which satsify

$$\begin{aligned}
&u\in\mathcal{C}^\infty_0(\overline{X};\mathfrak{f})\quad,\quad\gamma_{odd}u=0\,,i.e.\;\gamma u(q,-p)=j\gamma u(q,p) && (12)\\
\text{and}\quad&\partial_{q^1}u=\mathcal{O}(|q^1|^\infty)\,, && (13)
\end{aligned}$$

where the last line is the local writing in a coordinate systems such that $g=1\oplus^\perp m$.
Convention: We keep the letter P, often with additional informational indices, for differential operators acting on $\mathcal{C}^\infty$-functions or distributions. When P denotes a Kramers-Fokker-Planck operator, the letter K will be used for closed (and actually maximal accretive) realizations, also parametrized by A (and j), of P in $L^2(X,dqdp;\mathfrak{f})$.

1.3 Main results

Although the analysis mixes the two cases $A=0$ and $A\neq 0$, we separate them here for the sake of clarity.

Theorem 1.1. *Let Q be a riemannian compact manifold with boundary or a compact perturbation of the euclidean half-space $\overline{\mathbb{R}^d_-}=(-\infty,0]\times\mathbb{R}^{d-1}$. The operator $K_{\pm,0,g}-\frac{d}{2}=P_{\pm,g}-\frac{d}{2}$ with the domain defined by (9)(10)(11) with $A=0$ is maximal accretive.*
It satisfies the integration by part identity

$$\mathbb{R}\mathrm{e}\,\langle u\,,\,(K_{\pm,0,g}+\frac{d}{2})u\rangle=\langle u\,,\,(\frac{d}{2}+\mathcal{O}_g)u\rangle=\|u\|^2_{L^2(Q;\mathcal{H}^1)}$$

for all $u\in D(K_{\pm,0,g})$.
There exists $C>0$, such that

$$\begin{aligned}\langle\lambda\rangle^{1/4}\|u\|+\langle\lambda\rangle^{\frac{1}{8}}\|u\|_{L^2(Q;\mathcal{H}^1)}+\|u\|_{H^{\frac{1}{3}}(\overline{Q};\mathcal{H}^0)}&\\ +\langle\lambda\rangle^{\frac{1}{4}}\|(1+|p|_q)^{-1}\gamma u\|_{L^2(\partial X,|p_1|dq'dp;\mathfrak{f})}&\leq C\|(K_{\pm,0,g}-i\lambda)u\|\end{aligned}$$

for all $u\in D(K_{\pm,0,g})$ and all $\lambda\in\mathbb{R}$.
When $\Phi\in\mathcal{C}^\infty_b([0,+\infty))$, is such that $\Phi(0)=0$, there exist a constant $C'>0$ independent of Φ and constant $C_\Phi>0$ such that

$$\|\Phi(d_g(q,\partial Q))\mathcal{O}_g u\|\leq C'\|\Phi\|_{L^\infty}\|(K_{\pm,0,g}-i\lambda)u\|+C_\Phi\|u\|\,,$$

for all $u\in D(K_{\pm,0,g})$ and all $\lambda\in\mathbb{R}$.
The adjoint $K^*_{\pm,0,g}$ equals $K_{\mp,0,g}$.
Finally the set $\mathcal{D}(\overline{X},j)$ defined by (12)(13) is dense in $D(K_{\pm,0,g})$ endowed with its graph norm.

In the case when $A\neq 0$ and by assuming (5)(6)(7)(8), we have better estimates of the trace, but no density theorem in general.

Theorem 1.2. *Let Q be a riemannian compact manifold with boundary or a compact perturbation of the euclidean half-space $\overline{\mathbb{R}^d_-}=(-\infty,0]\times\mathbb{R}^{d-1}$. Assume that the operator $A\in\mathcal{L}(L^2(\partial X,|p_1|dq'dp;\mathfrak{f}))$ satisfies (5)(6)(7)(8). The operator $K_{\pm,A,g}-\frac{d}{2}=P_{\pm,g}-\frac{d}{2}$ with the domain defined by (9)(10)(11) is maximal accretive.*
It satisfies the integration by part identity

$$\mathbb{R}\mathrm{e}\,\langle u\,,\,(K_{\pm,A,g}+\frac{d}{2})u\rangle=\|u\|^2_{L^2(Q;\mathcal{H}^1)}+\mathbb{R}\mathrm{e}\,\langle\gamma_{ev}u\,,\,A\gamma_{ev}u\rangle_{L^2(\partial X,|p_1|dq'dp;\mathfrak{f})}\,,$$

for all $u \in D(K_{\pm,A,g})$ *.*
There exists $C > 0$ *and for any* $t \in [0, \frac{1}{18})$ *there exists* $C_t > 0$ *such that*

$$\langle\lambda\rangle^{1/4}\|u\| + \langle\lambda\rangle^{\frac{1}{8}}\|u\|_{L^2(Q;\mathcal{H}^1)} + C_t^{-1}\langle\lambda\rangle^{\frac{1}{8}}\|u\|_{H^t(\overline{Q};\mathcal{H}^0)} \\ + \langle\lambda\rangle^{\frac{1}{8}}\|\gamma u\|_{L^2(\partial X,|p_1|dq'dp;\mathfrak{f})} \leq C\|(K_{\pm,A,g} - i\lambda)u\|$$

for all $u \in D(K_{\pm,A,g})$ *and all* $\lambda \in \mathbb{R}$ *.*
When $\Phi \in \mathcal{C}_b^\infty([0,+\infty))$ *, is such that* $\Phi(0) = 0$ *, there exist a constant* $C' > 0$ *independent of* Φ *and constant* $C_\Phi > 0$ *such that*

$$\|\Phi(d_g(q,\partial Q))\mathcal{O}_g u\| \leq C'\|\Phi\|_{L^\infty}\|(K_{\pm,A,g} - i\lambda)u\| + C_\Phi\|u\|\,,$$

for all $u \in D(K_{\pm,A,g})$ *and all* $\lambda \in \mathbb{R}$ *.*
The adjoint $K^*_{\pm,A,g}$ *equals* $K_{\mp,A^*,g}$ *.*

Specific cases, in particular the one dimensional case and the flat multidimensional case, will be studied with weaker assumptions or stronger results. Other results are gathered in Section 8. Various applications are listed in Section 9.

1.4 Guidelines for reading this text

Although this text is rather long the strategy is really a classical one for boundary value problems (see e.g. [ChPi][BdM][HormIII]-Chap 20):

1. The first step (see especially [BdM]) consists in a full understanding of the simplest one-dimensional problem.

2. In the second step, separation of variable arguments are introduced in order to treat straight half-space problems.

3. The last step, is devoted to the local reduction of the general problem to the straight half-space problem, by checking that the correction terms due to the change of coordinates can be considered in some sense as perturbative terms.

Once this is said, one has to face two difficulties:

a) The one-dimensional boundary value problem is a two-dimensional problem with $(q^1, p_1) \in (-\infty, 0] \times \mathbb{R}$, with p_1-dependent coefficients. Moreover it rather looks like a corner problem because at $q^1 = 0$ the cases $p_1 > 0$ and $p_1 < 0$ are discontinuously partitionned. Singularities actually occur at $p_1 = 0$ and have to be handled with weighted L^2-norms.

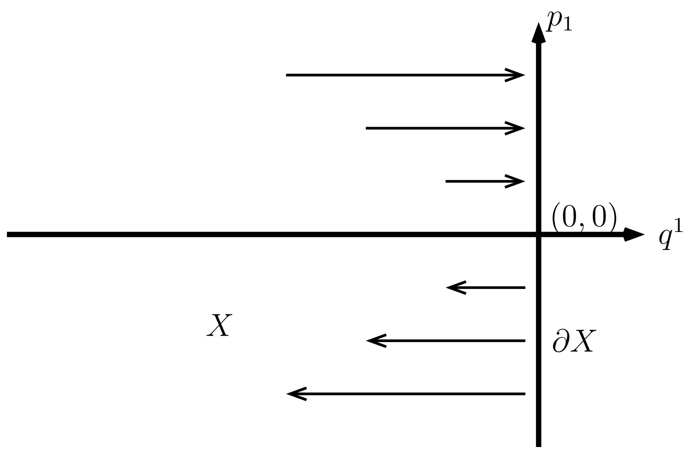

Fig.1: The boundary $\partial X = \{q^1 = 0\}$ and the vector field $p_1 \partial_{q^1}$ are represented. For the absorbing case, the boundary condition says $\gamma u(p_1) = 0$ for $p_1 < 0$ and corresponds to the case ($j = 1$ and $A = 1$).

b) In Step 3, the extrinsic curvature of the boundary brings a singular perturbation : after a change of coordinates the corresponding perturbative terms are not negligible as compared to the regularity and decay estimates obtained in Step 2. In the end, the subelliptic estimates are deteriorated by this curvature effect. Here is the geometric reason: When one considers the geodesic flow on $\overline{Q} = Q \sqcup \partial Q$, that is the flow of the hamiltonian vector field $\mathcal{Y}$ on T^*Q or S^*Q completed by specular reflection at the boundary, one has to face the well known problem of glancing rays. There are the two categories of gliding and grazing rays (see fig.2 below) which prevent from smooth symplectic reductions to half-space problems (no $\mathcal{C}^\infty$ solutions to the eikonal equations for example). Those problems have been widely studied with the propagation of singularities for the wave equation (see e.g.

[AnMe][Tay1][Tay2][MeSj1][MeSj2]). Here the question is: To what extent the dissipative term $\mathcal{O}_g = \frac{-\Delta_p + |p|_q^2}{2}$ and the hypoelliptic bracketing (with the hamiltonian vector field $\mathcal{Y}$) allow to absorb those regularity problems ?

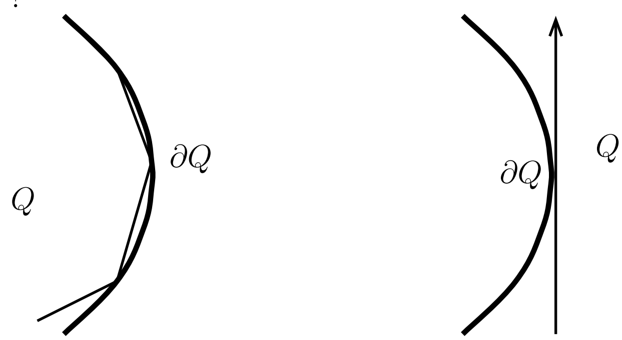

Fig.2: The left picture show a (approximately) gliding ray and the right one a grazing ray.

A deductive reading is possible starting from Section 2 to Section 9, by referring occasionally to the rather elementary arguments gathered in Appendix A and Appendix B. According to what the reader is looking for, below are some details about the various Sections.

- The reader who wants to know the consequences and applications of Theorem 1.1 and Theorem 1.2 can go directly to Section 8 and Section 9. Section 8 contains corollaries about the spectral properties of $K_{\pm,A,g}$ and the exponential decay properties of $e^{-tK_{\pm,A,g}}$. There are also extensions to the case when a potential $V(q)$ is added to the kinetic energy $\mathcal{E}(q,p) = \frac{|p|_q^2}{2}$, or to the case when $Q \times \mathfrak{f}$ is replaced by some fiber bundle. Section 9 lists various natural boundary condition operators A for the scalar case, with interpretations in terms of the Langevin stochastic process, and then considers specific cases for Bismut's hypoelliptic Laplacian.

- Appendix A recalls the maximal (i.e. with optimal exponent) subelliptic estimates in the translation invariant case. This is carried out with elementary arguments: Fourier transform, harmonic oscillator Hamiltonian $-\Delta_p + |p|^2$ and the one-dimensional complex Airy operator $-\partial_x^2 + ix$. For the freshman in subelliptic estimates, this can be a good starting point.

- Section 2 provides a thorough study of the one-dimensional problem $\overline{Q} = (-\infty, 0]$ endowed with the euclidean metric $g = (dq^1)^2$. The singularity at $p_1 = 0$ is solved by introducing a quantization S of the function $\mathrm{sign}\,(p_1)$ related to some kind of Fourier series in the p_1-variable. The “Fourier” basis is a set of eigenvectors for the compact self-adjoint operator $(\frac{1}{2} + \mathcal{O}_g)^{-1} p_1$ acting in $\mathcal{H}^1$. The interior problem $(p_1 \partial_{q^1} + \frac{1}{2} + \mathcal{O}_g) u = f$, the condition $\gamma u \in L^2(\mathbb{R}, |p_1| dp_1)$ as well as the boundary condtions $\gamma_{odd} u = \mathrm{sign}\,(p_1) A \gamma_{odd} u$ are trivialized in terms of those Fourier series. A Calderon projector is introduced and general boundary value problems can be studied.

- Section 3 gathers several functional analysis properties for semigroup generators which satisy some subelliptic estimates. Like in [HerNi] (see also [EcHa]) those generators have good resolvent estimates in some cuspidal domain of the complex plane, and they interpolate between the notion of sectorial operators and the one of general maximal accretive operators. We shall speak of “cuspidal semigroup” or “cuspidal operator” for the maximal accretive generator. Although this does not provide very good subelliptic estimates, this property is stable by tensorization, which allow separation of variable arguments for the straight half-space problem. Perturbative results are also provided.

- With Appendix A, Section 2 and Section 3 only, the reader will be convinced that subelliptic estimates are reasonable for boundary value problem for which the differential operator and the boundary conditions allow the separation of the variables (q^1, p_1) and (q', p') . This is reconsidered in Section 4 with the specific case $A = 0$ and $A = 1$. For $A = 1$ non homogeneous boundary value problems are studied with the help of the one-dimensional case and the variational argument proposed in kinetic theory (see [Car][Luc]) and relying on the weak formulations of [Lio]. For the case $A = 0$, the problem is extended to the whole space with the reflection $(q^1, p_1) \to (-q^1, -p_1)$. Both cases are useful and their nice properties compensate the fact that we do not have accurate enough information about the Calderon projector for Kramers-Fokker-Planck operators in $\overline{\mathbb{R}^{2d}_-}$ when $d > 1$.

- Half-space problems with a general boundary operator A are treated in Section 5 after a reduction to the boundary with the help of the case $A = 1$. General subelliptic estimates are obtained by using the reflection principle $(q^1, p_1) \to (-q^1, -p_1)$ and the nice properties of the case $A = 0$.

- The geometric analysis really starts in Section 6 where the maximal subelliptic estimates of Lebeau in [Leb2] are written in the way which is used afterwards. It concerns the case of manifolds without boundary, but the difficulty of a non vanishing curvature is recalled after [Leb1][Leb2] and explained in view of boundary value problems. Spatial partitions of unities and dyadic partitions of unities in the momentum variables are also introduced in this section. Repeatedly used easy formulas for partitions of unities are recalled in Appendix B.

- The most technical part is in Section 7: We study the case of cylinders $\overline{Q} = \times(-\infty, 0] \times Q'$ where Q' is compact and the metric g has the canonical form (2). Accurate subelliptic estimates for the case when $\partial_{q^1} g \equiv 0$ (straight cylinder) are deduced from the general functional framework of Section 5. With a dyadic partition of unity in the momentum p, the large momentum analysis is replaced by some kind of semiclassical asymptotics on a compact set in (q, p). The more general case $\partial_{q^1} g \neq 0$ is sent to the case $\partial_{q^1} g \equiv 0$ with the help of a non symplectic transformation on $X = T^*Q$ which is the identity along the boundary $\partial X = \{(q^1, q', p_1, p'), q^1 = 0\}$. A delicate use of the second resolvent formula for the semiclassical problem then allows to aborb the perturbation which encodes the curvature effect. After gluing all the dyadic pieces in p and then using a spatial partition of unity (the local form $A = A(q, |p|_q)$ is used for both steps) Theorem 1.1 and Theorem 1.2 are proved.

2 One dimensional model problem

As we learn from the general theory of boundary value problems for linear PDEs, relying on the construction of Calderon's projector, as presented in [ChPi][BdM][HormIII]-Chap 20, the key point is a good understanding of half-line one dimensional model problems with constant coefficients. Here the one dimensional case is actually a bidimensional problem with p-dependent coefficients. The main ingredient of this section is the introduction of adapted "Fourier series" in the p-variable which allow a thourough study of Calderon's projector and general boundary value problems.

2.1 Presentation

We consider the simple case when $\overline{Q} = \overline{\mathbb{R}_-} = (-\infty, 0]$ is endowed with the euclidean metric. The cotangent bundle is $X = T^*Q = \mathbb{R}^2_-$, $\overline{X} = T^*\overline{Q} = \overline{\mathbb{R}^2_-}$. The Kramers-Fokker-Planck operator is simply given by

$$\mathcal{Y}_{\mathcal{E}} = p\partial_q \quad , \quad \mathcal{O} = \frac{1}{2}(-\partial_p^2 + p^2)\,,$$

$$\text{and} \qquad P = p\partial_q + \frac{-\partial_p^2 + p^2}{2} \quad \text{on} \quad \mathbb{R}^2_- = \left\{(q,p) \in \mathbb{R}^2\,,\ q < 0\right\}\ .$$

The space $\mathcal{H}^s(q) = (\frac{1}{2} + \mathcal{O})^{-s/2} L^2(\mathbb{R}^d; \mathfrak{f})$ and its norm do not depend on $q \in \overline{\mathbb{R}_-}$ and we simply write $\mathcal{H}^s(q) = \mathcal{H}^s$. The Sobolev space $H^s(\overline{\mathbb{R}_-})$ is the usual one and the notations, $L^2(\mathbb{R}_-; \mathcal{H}^s)$ is better written here $L^2(\mathbb{R}_-, dq; \mathcal{H}^s)$ and $H^s(\overline{\mathbb{R}_-}; \mathcal{H}^s)$ keeps the same meaning as in the introduction.
We want to understand the boundary conditions along $\{q = 0\}$, which ensure the maximal accretivity of the associated closed operators. When $u \in \mathcal{S}(\overline{\mathbb{R}^2_-}; \mathfrak{f})$ one computes

$$\mathrm{Re}\,\langle u\,,\, Pu\rangle = \frac{1}{2}\int_{\mathbb{R}^2_-} |\partial_p u|^2_{\mathfrak{f}} + |pu|^2_{\mathfrak{f}}\ dqdp + \frac{1}{2}\int_{\mathbb{R}} p|u(0,p)|^2_{\mathfrak{f}}\ dp\,.$$

For $u \in \mathcal{S}(\overline{\mathbb{R}^2_-}; \mathfrak{f})$ (or possibly less regular u), a trace $\gamma u(p) = u(0,p)$ is decomposed according to the general definitions (3)(4) into

$$\gamma_{ev} u(p) = \frac{1}{2}\left[\gamma u(p) + j\gamma u(-p)\right] \qquad \text{and} \qquad \gamma_{odd} u(p) = \frac{1}{2}\left[\gamma u(p) - j\gamma u(-p)\right]\ ,$$

$$\gamma_+ u(p) = \gamma u(p) 1_{(0,+\infty)}(p) \qquad \text{and} \qquad \gamma_- u(p) = \gamma u(p) 1_{(-\infty,0)}(p)\ ,$$

where j is the unitary involution on $\mathfrak{f}$. Notice the relations

$$[\gamma_{ev} u + \mathrm{sign}\,(p)\gamma_{odd} u](p) = \gamma_+ u(p) + j(\gamma_+ u)(-p)\,,$$

$$\text{and} \qquad [\gamma_{ev} u - \mathrm{sign}\,(p)\gamma_{odd} u](p) = j(\gamma_- u)(-p) + \gamma_- u(p)\,.$$

With those notations the boundary term can be written

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}} p|u(0,p)|_{\mathfrak{f}}^2\, dp &= \frac{1}{2}\int_{\mathbb{R}}\left[|\gamma_+ u(p)|_{\mathfrak{f}}^2 - |\gamma_- u(p)|_{\mathfrak{f}}^2\right]\ |p|dp\\ &= \frac{1}{4}\int_{\mathbb{R}} |\gamma_{ev}u + \mathrm{sign}\,(p)\gamma_{odd}u|_{\mathfrak{f}}^2 - |\gamma_{ev}u - \mathrm{sign}\,(p)\gamma_{odd}u|_{\mathfrak{f}}^2\ |p|dp\\ &= \mathbb{R}\mathrm{e}\int_{\mathbb{R}}\langle \gamma_{ev}u(p)\,,\,\mathrm{sign}\,(p)\gamma_{odd}u(p)\rangle_{\mathfrak{f}}\ |p|dp\\ &= \mathbb{R}\mathrm{e}\,\langle\gamma_{ev}u\,,\,\mathrm{sign}\,(p)\gamma_{odd}u\rangle_{L^2(\mathbb{R},|p|dp;\mathfrak{f})}\,.\end{aligned}$$

A natural space for the traces is $L^2(\mathbb{R},|p|dp;\mathfrak{f})$ and when the boundary conditions are given by a linear relation of the form

$$\gamma_{odd}u = \mathrm{sign}\,(p)\times(A\gamma_{ev}u)\,,$$

the accretivity of the operator A is a necessary condition for having an accretive realization of P. We shall further assume that A is maximal accretive and commutes with the orthogonal projection $\Pi_{ev} : L^2(\mathbb{R},|p|dp;\mathfrak{f})\ni\gamma\mapsto \Pi_{ev}\gamma=\gamma_{ev}$. Then we consider the operator K_A given by

$$D(K_A)=\left\{u\in L^2(\mathbb{R}_-;\mathcal{H}^1)\,,\ Pu\in L^2(\mathbb{R}_-^2;\mathfrak{f})\,,\ \gamma_{odd}u=\mathrm{sign}\,(p)A\gamma_{ev}u\right\} \tag{14}$$

$$\forall u\in D(K_A)\,,\ K_Au=Pu=(p\partial_q+\tfrac{-\partial_p^2+p^2}{2})u\,. \tag{15}$$

Note that the conditions occuring in $D(K_A)$ are the minimal ones to give a meaning to the previous calculations.
Guided by the hyperbolic nature of $p\partial_q$, the kinetic theory (see [Bar][Car][Luc]) more often formulates the boundary conditions in term of γ_+u (outflow) and γ_-u (inflow). There are two fundamental examples with $j=\mathrm{Id}_{\mathfrak{f}}$

- Specular reflection: It is usually written $\gamma_-u=\gamma_+u$ and it reads now

$$\gamma_{odd}u=0\quad,\quad A=0\,.$$

- Absorbing boundary: It is usually written $\gamma_-u=0$ and it reads now

$$\gamma_{odd}u=\mathrm{sign}\,(p)\gamma_{ev}u\quad,\quad A=\mathrm{Id}.$$

Both example fulfill the maximal accretivity and commutation conditions.

2.2 Results

The general results hold here for possibly unbounded maximal accretive operators $(A, D(A))$ in $L^2(\mathbb{R}, |p|dp; \mathfrak{f})$. The commutation with Π_{ev} is then defined by $\Pi_{ev} e^{-tA} = e^{-tA}\Pi_{ev}$ for all $t \geq 0$.

Theorem 2.1. *Assume that $(A, D(A))$ is maximal accretive on $L^2(\mathbb{R}, |p|dp; \mathfrak{f})$ and commutes with Π_{ev} . Then the operator $K_A - \frac{1}{2}$ defined by (14)(15) is densely defined and maximal accretive in $L^2(\mathbb{R}^2_-, dqdp; \mathfrak{f})$.*
For every $u \in D(K_A)$, the traces $\gamma_{ev} u = \Pi_{ev}\gamma u$ and $\gamma_{odd} u = \Pi_{odd}\gamma u$ are well defined with

$$\|\gamma_{odd} u\|_{L^2(\mathbb{R},|p|dp;\mathfrak{f})} \leq C\|u\|_{D(K_A)} = C\left[\|u\| + \|K_A u\|\right] ,$$
$$\|\gamma_{ev} u\|_{L^2(\mathbb{R},|p|dp;\mathfrak{f})} + \|A\gamma_{ev} u\|_{L^2(\mathbb{R},|p|dp;\mathfrak{f})} = \|\gamma_{ev} u\|_{D(A)} \leq C\left[\|u\| + \|K_A u\|\right] .$$

Any $u \in D(K_A)$ satisfies the integration by part identity

$$\mathbb{R}\mathrm{e}\, \langle \gamma_{ev} u\, ,\, A\gamma_{ev} u\rangle_{L^2(\mathbb{R},|p|dp;\mathfrak{f})} + \|u\|^2_{L^2(\mathbb{R}_-,dq;\mathcal{H}^1)} = \mathbb{R}\mathrm{e}\, \langle u\, ,\, ((\frac{1}{2} + K_A)u\rangle . \quad (16)$$

The above result is completed by the two following propositions.

Proposition 2.2. *Under the hypothesis of Theorem 2.1 , the boundary value problem*

$$(P - z)u = f \quad , \quad \gamma_{odd} u = sign\,(p) A\gamma_{ev} u\, ,$$

admits a unique solution in $L^2(\mathbb{R}_-, dq; \mathcal{H}^1)$ when $f \in L^2(\mathbb{R}_-, dq; \mathcal{H}^{-1})$ and $\mathbb{R}\mathrm{e}\ z < \frac{1}{2}$. This solution belongs to $\mathcal{C}^0_b((-\infty, 0]; L^2(\mathbb{R}, |p|dp; \mathfrak{f}))$ and $\gamma_{ev} u \in D(A)$. This provides a unique continuous extension of the resolvent $(K_A - z)^{-1} : L^2(\mathbb{R}_-, dq; \mathcal{H}^{-1}) \to L^2(\mathbb{R}_-, dq; \mathcal{H}^1)$ when $\mathbb{R}\mathrm{e}\ z < \frac{1}{2}$.

Proposition 2.3. *Under the hypothesis of Theorem 2.1 , the adjoint of K_A^* of $(K_A, D(K_A))$ is given by*

$$D(K_A^*) = \left\{ u \in L^2(\mathbb{R}_-, dq; \mathcal{H}^1)\, ,\ \begin{array}{c} P_- u \in L^2(\mathbb{R}^2_-, dqdp; \mathfrak{f})\, , \\ \gamma_{odd} u = -\, sign\,(p) A^*\gamma_{ev} u \end{array} \right\} ,$$
$$\forall u \in D(K_A^*)\, , \quad K_A^* u = P_- u = (-p\partial_q + \frac{-\partial_p^2 + p^2}{2})u\, .$$

2.3 Fourier series in $\mathcal{H}^1$ and $L^2(\mathbb{R},|p|dp)$

Some results of this section will be stated with $\mathfrak{f}=\mathbb{C}$ and $j=\mathrm{Id}$. In general we shall use the orthogonal decomposition

$$\begin{aligned} & \mathfrak{f}=\mathfrak{f}_{ev}\overset{\perp}{\oplus}\mathfrak{f}_{odd}\,, \\ \text{with} \quad & \mathfrak{f}_{ev}=\ker(j-\mathrm{Id}) \quad , \quad \mathfrak{f}_{odd}=\ker(j+\mathrm{Id})\,. \end{aligned} \tag{17}$$

The harmonic oscillator hamiltonian, $\mathcal{O}=\frac{1}{2}(-\partial_p^2+p^2)\otimes\mathrm{Id}_{\mathfrak{f}}$ satisfies

$$\begin{aligned} & \mathcal{O}-\frac{1}{2}=a^*a\otimes\mathrm{Id}_{\mathfrak{f}} \quad , \quad \mathcal{O}+\frac{1}{2}=aa^*\otimes\mathrm{Id}_{\mathfrak{f}}=(1+a^*a)\otimes\mathrm{Id}_{\mathfrak{f}}\,, \\ \text{with} \quad & a=\frac{1}{\sqrt{2}}(\partial_p+p) \quad , \quad a^*=\frac{1}{\sqrt{2}}(-\partial_p+p)\,. \end{aligned}$$

The Hermite functions are normalized as

$$\varphi_0=\pi^{-1/4}e^{-\frac{p^2}{2}} \quad , \quad \varphi_n=(n!)^{-1/2}(a^*)^n\varphi_0\,,$$

and form an orthonormal family of $L^2(\mathbb{R},dp;\mathbb{C})$ of eigenvectors of a^*a. Note

$$\begin{aligned} & (\frac{1}{2}+\mathcal{O})^s=\sum_{n\in\mathbb{N}^*}n^s|\,(\varphi_{n-1}\rangle\langle\varphi_{n-1}|\otimes\mathrm{Id}_{\mathfrak{f}}) \\ \text{and} \quad & \|u\|_{\mathcal{H}^s}^2=\sum_{n\in\mathbb{N}^*}n^s|u_n|_{\mathfrak{f}}^2 \quad , \quad u(p)=\sum_{n\in\mathbb{N}^*}u_n\varphi_{n-1}(p)\,. \end{aligned}$$

While studying the maximal accretivity of K_A one has to study the equation

$$(\frac{1}{2}+P)u=f \quad , \quad \gamma_{odd}u=\mathrm{sign}\;p\times A\gamma_{ev}u \quad , \quad u\in L^2(\mathbb{R}_-,dq;\mathcal{H}^1)\,.$$

For $f\in L^2(\mathbb{R}_-,dq;\mathcal{H}^{-1})$ and by setting $f=(\frac{1}{2}+\mathcal{O})\check{f}$, it becomes

$$(\frac{1}{2}+\mathcal{O})^{-1}p\partial_q u=\check{f} \quad , \quad \gamma_{odd}u=\mathrm{sign}\;p\times A\gamma_{ev}u$$

with now $\check{f}\in L^2(\mathbb{R}_-,dq;\mathcal{H}^1)$. The clue is the spectral analysis of the operator $(\frac{1}{2}+\mathcal{O})^{-1}p$ in $\mathcal{H}^1$.

2.3.1 Spectral resolution of $(\frac{1}{2}+\mathcal{O})^{-1}p$ on $\mathcal{H}^1$

The first result diagonalizes $(\frac{1}{2}+\mathcal{O})^{-1}p$ when $\mathfrak{f}=\mathbb{C}$.

Proposition 2.4. *Assume $\mathfrak{f}=\mathbb{C}$. The operator $(\frac{1}{2}+\mathcal{O})^{-1}p=(1+a^*a)^{-1}p$ is self-adjoint and compact in $\mathcal{H}^1$. Its spectrum equals $\pm(2\mathbb{N}^*)^{-1/2}$ and all eigenvalues are simple. For $\nu\in\pm(2\mathbb{N}^*)^{-1/2}$, a normalized eigenvector can be chosen as*

$$e_\nu(p)=i^{\frac{sign\,(\nu)}{2\nu^2}}\nu\varphi_{[\frac{1}{2\nu^2}-1]}(p-\frac{1}{\nu})\,.$$

With this choice $e_\nu(-p)=e_{-\nu}(p)$ so that $e_\nu+e_{-\nu}$ is even and $e_\nu-e_{-\nu}$ is odd.

Proof. The operator $(\frac{1}{2}+\mathcal{O})^{1/2}:\mathcal{H}^1\to L^2(\mathbb{R},dp)$ is unitary and $(\frac{1}{2}+\mathcal{O})^{-1}p$ is unitarily equivalent to $(\frac{1}{2}+\mathcal{O})^{-1/2}p(\frac{1}{2}+\mathcal{O})^{-1/2}$. As an operator lying in $\mathrm{OpS}(\langle p,\eta\rangle^{-1},\frac{dp^2}{\langle p\rangle^2}+\frac{d\eta^2}{\langle\eta\rangle^2})$ (see [HormIII]-Chap 18), it is bounded and compact on $L^2(\mathbb{R},dp)$ and clearly self-adjoint. The spectral equation writes

$$(\frac{1}{2}+\mathcal{O})^{-1}pe=\nu e\quad\text{in }\mathcal{S}'(\mathbb{R})\,,$$

which is equivalent to ($\nu=0$ cannot happen)

$$\left[-\partial_p^2+(p-\frac{1}{\nu})^2+1-\frac{1}{\nu^2}\right]e=0\,.$$

After setting $e(p)=\varphi(p-\frac{1}{\nu})$, this gives $(aa^*-\frac{1}{2\nu^2})\varphi=0$ which is possible only when

$$\frac{1}{2\nu^2}=n\in1+\mathbb{N}=\mathbb{N}^*\quad\text{and}\quad\varphi\in\mathbb{C}\varphi_{n-1}\,.$$

For $\nu=\pm(2n)^{-1/2}$ with $n\in\mathbb{N}^*$, we take

$$e_\nu(p)=\frac{e^{i\alpha_\nu}}{\|\varphi_{n-1}(.-\frac{1}{\nu})\|_{\mathcal{H}^1}}\varphi_{n-1}(.-\frac{1}{\nu})\,,$$

with $\alpha_\nu\in\mathbb{R}$. The norm $\|\varphi_{n-1}(.-\frac{1}{\nu})\|_{\mathcal{H}^1}$ is computed by

$$\begin{aligned}\|\varphi_{n-1}(.-\frac{1}{\nu})\|^2_{\mathcal{H}^1}&=\langle\varphi_{n-1}(.-\frac{1}{\nu})\,,\,(\frac{1}{2}+\mathcal{O})\varphi_{n-1}(.-\frac{1}{\nu})\rangle\\&=\langle\varphi_{n-1}\,,\,\frac{(-\partial_p^2+(p+\frac{1}{\nu})^2+1)}{2}\varphi_{n-1}\rangle\\&=\langle\varphi_{n-1}\,,\,(\frac{1}{2}+\mathcal{O})\varphi_{n-1}\rangle+\frac{1}{2\nu^2}=\frac{1}{\nu^2}\,,\end{aligned}$$

where we have used $\langle \varphi_{n-1}\,,\, p\varphi_{n-1}\rangle = 0$. Therefore

$$\frac{1}{\|\varphi_{n-1}(.-\frac{1}{\nu})\|_{\mathcal{H}^1}} = |\nu|\,,$$

and we can take

$$e_\nu(p) = e^{i\beta_\nu}\nu\varphi_{n-1}(p-\frac{1}{\nu})\,,\quad \beta_\nu \in \mathbb{R}\,.$$

If one wants the two functions $e_\nu + e_{-\nu}$ and $e_\nu - e_{-\nu}$ to be even or odd, the equality $e_\nu(-p) = e_{-\nu}(p)$ enforces

$$\beta_\nu - \beta_{-\nu} = n\pi \qquad \mod (2\pi)\,,$$

and $\beta_\nu = \mathrm{sign}\,(\nu)\frac{\pi}{4\nu^2} \mod (2\pi)$ works. $\square$

With this spectral resolution one can define the following objects for general $(\mathfrak{f}, j)$.

Definition 2.5. *The self-adjoint operator $(\frac{1}{2}+\mathcal{O})^{-1}p = [(1+a^*a)^{-1}p]\otimes Id_{\mathfrak{f}}$ in $\mathcal{H}^1$ is denoted by A_0.*
For $s \in \mathbb{R}$, the Hilbert space $\mathcal{D}_s = A_0^s\mathcal{H}^1$, equivalently defined by

$$\mathcal{D}_s = \left\{ u = \sum_{\nu\in\pm(2\mathbb{N}^*)^{-1/2}} e_\nu \otimes u_\nu\,,\ \sum_{\nu\in\pm(2\mathbb{N}^*)^{-1/2}} |\nu|^{-2s}|u_\nu|_{\mathfrak{f}}^2 < +\infty \right\}\,,$$

is endowed with the scalar product

$$\langle u\,,\, u'\rangle_{\mathcal{D}_s} = \sum_{\nu\in\pm(2\mathbb{N}^*)^{-1/2}} |\nu|^{-2s}\langle u_\nu\,,\, u'_\nu\rangle_{\mathfrak{f}}\,.$$

The space $\mathcal{D}_\infty = \cap_{s\in\mathbb{R}}\mathcal{D}_s$ is a Fréchet space with the family of norms $(\|\ \|_{\mathcal{D}_n})_{n\in\mathbb{N}}$ dense in $\mathcal{H}^1$ and its dual is $\mathcal{D}_{-\infty} = \cup_{s\in\mathbb{R}}\mathcal{D}_s$.
In those spaces, the operator S is defined by

$$S\left(\sum_{\nu\in\pm(2\mathbb{N}^*)^{-1/2}} e_\nu\otimes u_\nu\right) = \sum_{\nu\in\pm(2\mathbb{N}^*)^{-1/2}} sign\,(\nu) e_\nu\otimes u_\nu\,.$$

For $\nu \in (2\mathbb{N}^*)^{-1/2}$, the following notations will be used

$$
\begin{aligned}
&V_\nu = \left(\mathbb{C}e_\nu \overset{\perp}{\oplus} \mathbb{C}e_{-\nu}\right) \otimes \mathfrak{f} = V_{\nu,ev} \overset{\perp}{\oplus} V_{\nu,odd}\,,\\
&e_{\nu,ev} = \frac{1}{\sqrt{2}}(e_\nu + e_{-\nu}) \quad , \quad e_{\nu,odd} = \frac{1}{\sqrt{2}}(e_\nu - e_{-\nu})\,,\\
&V_{\nu,ev} = (\mathbb{C}e_{\nu,ev}) \otimes \mathfrak{f}_{ev} \overset{\perp}{\oplus} (\mathbb{C}e_{\nu,odd}) \otimes \mathfrak{f}_{odd}\,,\\
&V_{\nu,odd} = (\mathbb{C}e_{\nu,odd}) \otimes \mathfrak{f}_{ev} \overset{\perp}{\oplus} (\mathbb{C}e_{\nu,ev}) \otimes \mathfrak{f}_{odd}\,,
\end{aligned}
$$

where $\mathfrak{f}_{ev,odd}$ are defined by (17).

Here is a list of obvious properties

- $\mathcal{D}_s = \bigoplus^{\perp}_{\nu \in (2\mathbb{N}^*)^{-1/2}} V_\nu$ and $\|u\|^2_{\mathcal{D}_s} = \sum_{\nu \in (2\mathbb{N}^*)^{-1/2}} \nu^{-2s}(|u_\nu|^2_{\mathfrak{f}} + |u_{-\nu}|^2_{\mathfrak{f}})$.
- S is a unitary self-adjoint operator in any $\mathcal{D}_s$ such that $S^2 = S$. The orthogonal projections $\frac{1+S}{2} = 1_{\mathbb{R}_+}(S)$ and $\frac{1-S}{2} = 1_{\mathbb{R}_-}(S)$ are given by

$$
\begin{aligned}
\frac{1+S}{2}\Big(\sum_{\nu \in \pm(2\mathbb{N}^*)^{-\frac{1}{2}}} e_\nu \otimes u_\nu\Big) &= \sum_{\nu \in +(2\mathbb{N}^*)^{-\frac{1}{2}}} e_\nu \otimes u_\nu\,,\\
\frac{1-S}{2}\Big(\sum_{\nu \in \pm(2\mathbb{N}^*)^{-\frac{1}{2}}} e_\nu \otimes u_\nu\Big) &= \sum_{\nu \in -(2\mathbb{N}^*)^{-\frac{1}{2}}} e_\nu \otimes u_\nu\,.
\end{aligned}
$$

  Note also the relations

$$
S(e_{\nu,ev} \otimes u_\nu) = e_{\nu,odd} \otimes u_\nu \quad , \quad S(e_{\nu,odd} \otimes u_\nu) = e_{\nu,ev} \otimes u_\nu\,. \tag{18}
$$

- With $\mathcal{D}_0 = \mathcal{H}^1$, $\mathcal{D}_s = D(|A_0|^{-s})$ when $s > 0$ while, for $s < 0$, $\mathcal{D}_s$ is the completion of $\mathcal{D}_0 = \mathcal{H}^1$ (or $\mathcal{S}(\mathbb{R})$) for the norm $\|u\|_{\mathcal{D}_s} = \||A_0|^{-s}u\|_{\mathcal{H}^1}$. In particular $\mathcal{D}_{-1}$ is the completion of $\mathcal{S}(\mathbb{R})$ for the norm

$$
\||A_0|u\|_{\mathcal{H}^1} = \|A_0 u\|_{\mathcal{H}^1} = \|(\frac{1}{2} + \mathcal{O})pu\|_{\mathcal{H}^1} = \|pu\|_{\mathcal{H}^{-1}}\,.
$$

  Note that $\mathcal{D}_{-1}$ is not embedded in $\mathcal{D}'(\mathbb{R})$, only in $\mathcal{D}'(\mathbb{R} \setminus \{0\})$, for it contains functions behaving like $\frac{1}{p}$ around $p = 0$.

- When u belongs to $\mathcal{D}_s \subset \mathcal{D}'(\mathbb{R}\setminus\{0\})$ for some $s\in\mathbb{R}$, the even (resp. odd) part, $\Pi_{ev}u = \frac{1}{2}(u(p)+ju(-p))$ (resp. $\Pi_{odd}u = \frac{1}{2}(u(p)-ju(-p))$), is the orthogonal projection of u onto $\oplus^{\perp}_{\nu\in(2\mathbb{N}^*)^{-1/2}} V_{\nu,ev}$ (resp. $\oplus^{\perp}_{\nu\in(2\mathbb{N}^*)^{-1/2}} V_{\nu,odd}$). More precisely, for

  $$u = \sum_{\nu\in\pm(2\mathbb{N}^*)^{-1/2}} e_\nu\otimes u_\nu = \sum_{\nu\in\pm(2\mathbb{N}^*)^{-1/2}} e_\nu\otimes(\frac{1+j}{2}u_\nu + \frac{1-j}{2}u_\nu)\,,$$
  $$\text{with}\quad \frac{1+j}{2}u_\nu\in\mathfrak{f}_{ev}\quad,\quad \frac{1-j}{2}u_\nu\in\mathfrak{f}_{odd}\,,$$

  the orthogonal decomposition is

  $$\begin{aligned} u = & \underbrace{\sum_{\nu>0} e_{\nu,ev}\otimes(\frac{1+j}{2})\frac{u_\nu+u_{-\nu}}{\sqrt{2}} + e_{\nu,odd}\otimes(\frac{1-j}{2})\frac{u_\nu-u_{-\nu}}{\sqrt{2}}}_{\Pi_{ev}u} \\ & + \underbrace{\sum_{\nu>0} e_{\nu,odd}\otimes(\frac{1+j}{2})\frac{u_\nu-u_{-\nu}}{\sqrt{2}} + e_{\nu,ev}\otimes(\frac{1-j}{2})\frac{u_\nu+u_{-\nu}}{\sqrt{2}}}_{\Pi_{odd}u}\,. \end{aligned} \tag{19}$$

  This formula and (18) imply $S\Pi_{ev} = \Pi_{odd}S$.

- For $u, u'\in\mathcal{S}(\mathbb{R};\mathfrak{f})$, the following identites hold:

  $$\begin{aligned}\langle u\,,\,\mathrm{sign}\,(p)u'\rangle_{L^2(\mathbb{R},|p|dp;\mathfrak{f})} = \langle u\,,\,pu'\rangle &= \langle u\,,\,(\frac{1}{2}+\mathcal{O})(\frac{1}{2}+\mathcal{O})^{-1}pu'\rangle \\ &= \langle u\,,\,A_0u'\rangle_{\mathcal{H}^1} = \langle u\,,\,Su\rangle_{\mathcal{D}_{-\frac{1}{2}}}\,.\end{aligned}$$

2.3.2 An interpolation result

The main result of this section is the

Proposition 2.6. *The interpolated Hilbert space $\mathcal{D}_{-\frac{1}{2}} = [\mathcal{H}^1\,,\,A_0^{-1}\mathcal{H}^1]_{\frac{1}{2}}$ is nothing but*

$$\mathcal{D}_{-\frac{1}{2}} = L^2(\mathbb{R},|p|dp;\mathfrak{f})\,.$$

Moreover there is a bounded invertible positive operator M on $\mathcal{D}_{-\frac{1}{2}}$ (which has the same properties on $L^2(\mathbb{R},|p|dp)$) such that

$$\begin{aligned}&\forall u,u'\in L^2(\mathbb{R},|p|dp;\mathfrak{f})\,,\quad \langle u\,,\,u'\rangle_{L^2(\mathbb{R},|p|dp;\mathfrak{f})} = \langle u\,,\,Mu'\rangle_{\mathcal{D}_{-\frac{1}{2}}}\,,\\ &\forall u,u'\in\mathcal{D}_{-\frac{1}{2}}\,,\quad \langle u\,,\,u'\rangle_{\mathcal{D}_{-\frac{1}{2}}} = \langle u\,,\,M^{-1}u'\rangle_{L^2(\mathbb{R},|p|dp;\mathfrak{f})}\,,\\ &S = M\circ \mathit{sign}\;p = \mathit{sign}\;p\circ M^{-1}\,.\end{aligned}$$

The projections Π_{ev}, Π_{odd} given by $(\Pi_{ev}u)(p) = \frac{1}{2}[u(p)+ju(-p)]$ and $(\Pi_{odd})u(p) = \frac{1}{2}[u(p)-ju(-p)]$ are orthogonal for both scalar products and commute with M .

Proof. By functional calculus the Hilbert space $\mathcal{D}_{-\frac{1}{2}}$ is the interpolation (complex or real interpolation are equivalent in this case see [BeLo]) of $[\mathcal{D}_0, \mathcal{D}_{-1}]_{\frac{1}{2}}$. Moreover $\mathcal{D}_0 = \mathcal{H}^1$ while $\mathcal{D}_{-1} = A_0^{-1}\mathcal{H}^1$ is the completion of $\mathcal{S}(\mathbb{R};\mathfrak{f})$ for the norm $\|p.\|_{\mathcal{H}^{-1}}$.
a) Consider the multiplication operator $u \mapsto \frac{p}{\sqrt{1+p^2}}u$. It belongs to $\mathrm{OpS}(1, \frac{dp^2}{\langle p\rangle^2}+\frac{d\eta^2}{\langle\eta\rangle^2})$ and $(\frac{1}{2}+\mathcal{O})^{\frac{1}{2}}(\frac{p}{\sqrt{1+p^2}}\times)(\frac{1}{2}+\mathcal{O})^{-\frac{1}{2}}$ belongs to $\mathrm{OpS}(1, \frac{dp^2}{\langle p\rangle^2}+\frac{d\eta^2}{\langle\eta\rangle^2})$. We deduce

$$\|\frac{p}{\sqrt{1+p^2}}u\|_{\mathcal{H}^1} \leq C\|u\|_{\mathcal{H}^1} \leq C\|u\|_{\mathcal{D}_0}\,,$$

for all $u \in \mathcal{H}^1$.
Similarly $(\frac{1}{2}+\mathcal{O})^{-\frac{1}{2}}\frac{1}{\sqrt{1+p^2}}(\frac{1}{2}+\mathcal{O})^{\frac{1}{2}} \in \mathrm{OpS}(1, \frac{dp^2}{\langle p\rangle^2}+\frac{d\eta^2}{\langle\eta\rangle^2})$ implies

$$\begin{aligned}\|\frac{p}{\sqrt{1+p^2}}u\|_{\mathcal{H}^{-1}} &\leq \|(\frac{1}{2}+\mathcal{O})^{-\frac{1}{2}}\left(\frac{1}{\sqrt{1+p^2}}\times\right)(\frac{1}{2}+\mathcal{O})^{\frac{1}{2}}(\frac{1}{2}+\mathcal{O})^{-\frac{1}{2}}pu\| \\ &\leq C\|(\frac{1}{2}+\mathcal{O})^{-\frac{1}{2}}pu\| \leq C\|pu\|_{\mathcal{H}^{-1}} \leq C\|u\|_{\mathcal{D}_{-1}}\,,\end{aligned}$$

for all $u \in \mathcal{S}(\mathbb{R};\mathfrak{f})$ and by density for all $u \in \mathcal{D}_{-1}$. By interpolation the mapping $\Phi_2 : u \mapsto \frac{p}{\sqrt{1+p^2}}u$ is continuous from $\mathcal{D}_{-\frac{1}{2}} = [\mathcal{D}_0\,,\,\mathcal{D}_{-1}]_{\frac{1}{2}}$ into $[\mathcal{H}^1\,,\,\mathcal{H}^{-1}]_{\frac{1}{2}} = L^2(\mathbb{R}, dp;\mathfrak{f})$ with norm less than C .
b) Consider the multiplication operator $u \mapsto \frac{1}{\sqrt{1+p^2}}u$. With the same arguments as above, which say now that $(\frac{1}{2}+\mathcal{O})^{\frac{1}{2}}(\frac{1}{\sqrt{1+p^2}}\times)(\frac{1}{2}+\mathcal{O})^{-\frac{1}{2}}$ and $(\frac{1}{2}+\mathcal{O})^{-\frac{1}{2}}(\frac{1}{\sqrt{1+p^2}}\times)(\frac{1}{2}+\mathcal{O})^{\frac{1}{2}}$ belong to $\mathrm{OpS}(1, \frac{dp^2}{\langle p\rangle^2}+\frac{d\eta^2}{\langle\eta\rangle^2})$, we deduce

$$\|\frac{1}{\sqrt{1+p^2}}u\|_{\mathcal{D}_0} = \|\frac{1}{\sqrt{1+p^2}}u\|_{\mathcal{H}^1} \leq C'\|u\|_{\mathcal{H}^1}\ ,\ \forall u \in \mathcal{H}^1\,,$$

$$\text{and }\|\frac{1}{\sqrt{1+p^2}}u\|_{\mathcal{D}_{-1}} = \|(\frac{1}{2}+\mathcal{O})^{-1/2}\frac{1}{\sqrt{1+p^2}}u\| \leq C'\|u\|_{\mathcal{H}^{-1}}\ ,\ \forall u \in \mathcal{H}^{-1}\,.$$

By interpolation the multiplication operator $\Phi_1 : u \mapsto \frac{1}{\sqrt{1+p^2}}u$ is continuous from $[\mathcal{H}^1\,,\,\mathcal{H}^{-1}]_{\frac{1}{2}} = L^2(\mathbb{R}, dp;\mathfrak{f})$ into $\mathcal{D}_{-\frac{1}{2}}$.

c) The mapping $\Phi_1 : L^2(\mathbb{R}, dp; \mathfrak{f}) \to \mathcal{D}_{-\frac{1}{2}}$ is one to one. Indeed assume $\Phi_1(u) = 0$ in $\mathcal{D}_{-\frac{1}{2}}$. Then $\Phi_2 \circ \Phi_1(u) = \frac{p}{1+p^2} u = 0$ in $L^2(\mathbb{R}, dp; \mathfrak{f})$, which implies $u = 0$ in $L^2(\mathbb{R}, dp; \mathfrak{f})$.
d) The mapping $\Phi_2 : \mathcal{D}_{-\frac{1}{2}} \to L^2(\mathbb{R}, dp; \mathfrak{f})$ is one to one. The proof is a little more delicate than c). For $\varphi \in \mathcal{S}(\mathbb{R}; \mathfrak{f})$ the equalities

$$\langle \varphi \, , \, Su \rangle_{\mathcal{D}_{-\frac{1}{2}}} = \int_{\mathbb{R}} \langle \sqrt{1+p^2} \varphi(p) \, , \, \frac{p}{\sqrt{1+p^2}} u(p) \rangle_{\mathfrak{f}} \; dp = \langle \sqrt{1+p^2} \varphi \, , \, \Phi_2 u \rangle \tag{20}$$

holds for any $u \in \mathcal{S}(\mathbb{R}; \mathfrak{f})$ and by density for any $u \in \mathcal{D}_{-\frac{1}{2}}$. If $\Phi_2(u) = 0$ in $L^2(\mathbb{R}, dp; \mathfrak{f})$ for some $u \in \mathcal{D}_{-\frac{1}{2}}$, the previous identity gives

$$\forall \varphi \in \mathcal{S}(\mathbb{R}; \mathfrak{f}) \, , \quad \langle \varphi \, , \, Su \rangle_{\mathcal{D}_{-\frac{1}{2}}} = 0 \, ,$$

which implies $Su = 0$ and therefore $u = 0$.
e) The identity (20) holds for $\varphi = \Phi_1 u_1 = \frac{1}{\sqrt{1+p^2}} u_1$ with $u_1 \in \mathcal{S}(\mathbb{R}; \mathfrak{f})$ and $u = u_2 \in \mathcal{D}_{-\frac{1}{2}}$. We obtain

$$\langle \Phi_1 u_1 \, , \, S u_2 \rangle_{\mathcal{D}_{-\frac{1}{2}}} = \langle \sqrt{1+p^2} \, \Phi_1 u_1 \, , \, \Phi_2 u_2 \rangle = \langle u_1 \, , \, \Phi_2 u_2 \rangle$$

which extends by continuity to any $u_1 \in L^2(\mathbb{R}, dp; \mathfrak{f})$ and any $u_2 \in \mathcal{D}_{-\frac{1}{2}}$. We deduce

$$\Phi_1^* = \Phi_2 S \quad \text{and} \quad \Phi_2^* = S \Phi_1 \, . \tag{21}$$

f) Call $\tilde{\Phi}_1$ (resp. $\tilde{\Phi}_2$) the bijection from $L^2(\mathbb{R}, dp; \mathfrak{f})$ onto $L^2(\mathbb{R}, (1+p^2)dp; \mathfrak{f})$ (resp. from $L^2(\mathbb{R}, \frac{p^2}{1+p^2} dp; \mathfrak{f})$ onto $L^2(\mathbb{R}, dp; \mathfrak{f})$) given by $\tilde{\Phi}_1 u = \frac{1}{\sqrt{1+p^2}} u$ (resp. $\tilde{\Phi}_2 u = \frac{|p|}{\sqrt{1+p^2}} u$). One has

$$\Phi_1 = j_1 \circ \tilde{\Phi}_1 \quad \text{and} \quad \Phi_2 = \tilde{\Phi}_2 \circ (\mathrm{sign}\,(p) \times) \circ j_2 \, ,$$

with $\quad j_1 u_1 = u_1$ for $u_1 \in L^2(\mathbb{R}, (1+p^2)dp; \mathfrak{f}) \quad$ and $\quad j_2 u_2 = u_2$ for $u_2 \in \mathcal{D}_{-\frac{1}{2}}$.

Moreover the equality

$$\langle j_1 \tilde{\Phi}_1 u_1 \, , \, S u_2 \rangle_{\mathcal{D}_{-\frac{1}{2}}} = \langle u_1 \, , \, \tilde{\Phi}_2 \, \mathrm{sign}\,(p) j_2 u_2 \rangle$$

valid for any $u_1, u_2 \in \mathcal{S}(\mathbb{R};\mathfrak{f})$ can be written, with $u_1' = \tilde{\Phi}_1(u_1) = \frac{u_1}{\sqrt{1+p^2}}$,

$$\begin{aligned}
\langle Sj_1u_1' \,,\, u_2\rangle_{\mathcal{D}_{-\frac{1}{2}}} &= \langle u_1 \,,\, \tilde{\Phi}_2 \,\mathrm{sign}\,(p) j_2 u_2\rangle \\
&= \int_{\mathbb{R}} \langle \sqrt{1+p^2}\, u_1'(p) \,,\, \frac{|p|}{\sqrt{1+p^2}}\,\mathrm{sign}\,(p)(j_2u_2)(p)\rangle_{\mathfrak{f}} \;dp \\
&= \int_{\mathbb{R}} \langle u_1'(p) \,,\, \mathrm{sign}\,(p)(j_2u_2)(p)\rangle_{\mathfrak{f}} \;|p| dp\,.
\end{aligned}$$

where $\int_{\mathbb{R}}\langle u(p) \,,\, v(p)\rangle_{\mathfrak{f}}\;|p|dp$ can be interpreted as a duality product between $L^2(\mathbb{R},(1+p^2)dp;\mathfrak{f})$ and $L^2(\mathbb{R},\frac{p^2}{1+p^2}dp;\mathfrak{f})$. After identifying $j_k u_k = u_k$ (and dropping the $'$), it is better written

$$\begin{aligned}
\langle Su_1 \,,\, u_2\rangle_{\mathcal{D}_{-\frac{1}{2}}} = \int_{\mathbb{R}} &\langle u_1 \,,\, (\mathrm{sign}\,(p) u_2\rangle_{\mathfrak{f}} \;|p| dp \\
&= \langle u_1 \,,\, \mathrm{sign}\,(p) u_2\rangle_{L^2(\mathbb{R},(1+|p|^2)dp;\mathfrak{f})\,,\, L^2(\mathbb{R},\frac{p^2}{1+p^2}dp,\mathfrak{f})}\,,
\end{aligned} \tag{22}$$

for all $u_1 \in L^2(\mathbb{R},(1+p^2)dp;\mathfrak{f}) \subset \mathcal{D}_{-\frac{1}{2}}$ and all $u_2 \in \mathcal{D}_{-\frac{1}{2}}$. This implies that

$$L^2(\mathbb{R},(1+p^2)dp;\mathfrak{f}) \xrightarrow{Sj_1} \mathcal{D}_{-\frac{1}{2}} \overset{\mathrm{sign}\,(p)j_2=(Sj_1)^*}{\longrightarrow} L^2(\mathbb{R},\frac{p^2}{1+p^2}dp;\mathfrak{f})$$

is a Hilbert triple. With the closed quadratic form $q(u) = \|u\|^2_{L^2(\mathbb{R},(1+p^2)dp;\mathfrak{f})} + \|u\|^2_{\mathcal{D}_{-\frac{1}{2}}}$, we can associate a non negative self-adjoint operator $B_0 \geq 1$ such that $L^2(\mathbb{R},(1+p^2)dp;\mathfrak{f}) = B_0^{-1/2}\mathcal{D}_{-\frac{1}{2}}$ and $L^2(\mathbb{R},\frac{p^2}{1+p^2}dp;\mathfrak{f}) = B_0^{1/2}\mathcal{D}_{-\frac{1}{2}}$. By the functional calculus for B_0 and complex interpolation, we deduce

$$\mathcal{D}_{-\frac{1}{2}} = \left[L^2(\mathbb{R},(1+p^2)dp;\mathfrak{f})\,,\, L^2(\mathbb{R},\frac{p^2}{1+p^2}dp;\mathfrak{f})\right]_{\frac{1}{2}} = L^2(\mathbb{R},|p|dp;\mathfrak{f})\,,$$

with equivalent Hilbert norms.

g) Let us specify the equivalence of the Hilbert scalar products. The Hilbert space $\mathcal{D}_{-\frac{1}{2}} = L^2(\mathbb{R},|p|dp;\mathfrak{f})$ is now endowed with two scalar products

$$\langle u_1 \,,\, u_2\rangle_1 = \langle u_1 \,,\, u_2\rangle_{\mathcal{D}_{-\frac{1}{2}}} \quad \text{and} \quad \langle u_1 \,,\, u_2\rangle_2 = \int_{\mathbb{R}} \langle u_1(p) \,,\, u_2(p)\rangle_{\mathfrak{f}}\;|p|dp\,.$$

The identity (22) implies now

$$\forall u_1, u_2 \in \mathcal{D}_{-\frac{1}{2}}\,, \quad \langle Su_1 \,, u_2\rangle_1 = \langle u_1 \,,\, \mathrm{sign}\,(p) u_2\rangle_2\,,$$

where S is unitary for $\langle\ ,\ \rangle_1$ and $\mathrm{sign}\,(p)$ is unitary for $\langle\ ,\ \rangle_2$. We deduce that $M = \mathrm{sign}\,(p)^{*1}S$ and $M' = S^{*2}\,\mathrm{sign}\,(p)$ are positive bounded operators (for the respective scalar products) such that

$$\langle u_1\,, Mu_2\rangle_1 = \langle u_1\,,\, u_2\rangle_2 \quad \text{and} \quad \langle u_1\,,\, u_2\rangle_1 = \langle u_1\,,\, M'u_2\rangle_2\,.$$

Hence $M' = M^{-1}$ and M is a bounded invertible positive operator for both scalar products.
The writing

$$\langle u_1\,,\, Su_2\rangle_1 = \langle Su_1\,,\, u_2\rangle = \langle u_1\,,\, \mathrm{sign}\,(p)u_2\rangle_2 = \langle u_1\,,\, M\,\mathrm{sign}\,(p)u_2\rangle_1\,,$$

implies $S = M\,\mathrm{sign}\,(p)$ which can be combined with the relations $S = S^{-1} = S^{*1}$ and $\mathrm{sign}\,(p) = \mathrm{sign}\,(p)^{-1} = \mathrm{sign}\,(p)^{*2}$.
h) It remains to prove the orthogonality of $\Pi_{ev}(u)$ and $\Pi_{odd}(u)$ for both scalar products and the commutation with M. For the second one it results from $\int_{\mathbb{R}} f(p)|p|dp = 0$ for every odd element of $L^1(\mathbb{R}, |p|dp)$. For the first scalar product, we have already checked in (19) that Π_{ev} and Π_{odd} are orthogonal projections in $\mathcal{D}^s$. The relation $\mathrm{sign}\,(p) \circ \Pi_{ev} = \Pi_{odd} \circ \mathrm{sign}\,(p)$ comes from the definition (3)(4) while (19) led to $S \circ \Pi_{ev} = \Pi_{odd} \circ S$. We obtain

$$M \circ \Pi_{ev} = S \circ \mathrm{sign}\,(p) \circ \Pi_{ev} = S \circ \Pi_{odd} \circ \mathrm{sign}\,(p) = \Pi_{ev} \circ S \circ \mathrm{sign}\,(p) = \Pi_{ev} \circ M\,.$$

□

We end this section with an additional lemma, related with the previous result: It ensures that some maximal accretivity is preserved when the scalar product is changed.

Lemma 2.7. *Consider a complex Hilbert space $\mathcal{D}$ endowed with two equivalent scalar products $\langle\ ,\ \rangle_1$ and $\langle\ ,\ \rangle_2$, like in Proposition 2.6, and let M be a positive bounded invertible operator (in $(\mathcal{D}, \langle\ ,\ \rangle_1)$ or $(\mathcal{D}, \langle\ ,\ \rangle_2)$) such that*

$$\forall u, u' \in \mathcal{D}\,, \quad \langle u\,,\, Mu'\rangle_1 = \langle u\,,\, u'\rangle_2\,.$$

Then a densely defined operator $(A, D(A))$ is maximal accretive in $(\mathcal{D}, \langle\ ,\ \rangle_2)$ iff $(MA, D(A))$ is maximal accretive in $(\mathcal{D}, \langle\ ,\ \rangle_1)$.

Proof. Assume $(A, D(A))$ maximal accretive in $(\mathcal{D}, \langle\ ,\ \rangle_2)$.
a) Then $(MA, D(A))$ is clearly accretive owing to

$$\forall u \in D(A)\,, \quad \mathrm{Re}\,\langle u\,,\, MAu\rangle_1 = \mathrm{Re}\,\langle u\,,\, Au\rangle_2 \geq 0\,.$$

b) Let us check that $\mathrm{Id} + MA$ is invertible. Our assumptions ensure that $\varepsilon\mathrm{Id} + A$ is invertible for $\varepsilon > 0$ in $\mathcal{D}$ and therefore it is the case also for $\varepsilon M + MA$. Hence $\varepsilon M + MA$ which is accretive in $(\mathcal{D}, \langle\ ,\ \rangle_1)$ is maximal accretive. Therefore $B_\varepsilon = \mathrm{Id} + \varepsilon M + MA$ with $D(B_\varepsilon) = D(A)$ is invertible for $\varepsilon > 0$ with the estimate

$$\forall u \in D(A)\,, \quad \|u\|_1 \|(\mathrm{Id}+\varepsilon M+MA)\|_1 \geq \mathrm{Re}\,\langle u\,,\,(\mathrm{Id}+\varepsilon M+MA)u\rangle_1 \geq \|u\|_1^2\,,$$

which means

$$\|B_\varepsilon^{-1}\|_{\mathcal{L}(\mathcal{D},\|\ \|_1)} \leq 1\,.$$

The second resolvent formula

$$(\mathrm{Id} + MA)^{-1} = B_\varepsilon^{-1}\left[\mathrm{Id} - \varepsilon M B_\varepsilon^{-1}\right]^{-1}\,,$$

and choosing $\varepsilon < \|M\|_{\mathcal{L}(\mathcal{D},\|\ \|_1)}^{-1}$ imply that $\mathrm{Id}+MA$ is invertible with $\mathrm{Ran}\,(I+MA)^{-1} = D(B_\varepsilon) = D(A)$.

The two points *a)* and *b)* prove the maximal accretivity of MA in $(\mathcal{D}, \langle\ ,\ \rangle_1)$ when $(A, D(A))$ is maximal accretive in $(\mathcal{D}, \langle\ ,\ \rangle_2)$. The converse statement comes from $A = M^{-1}MA$. $\square$

2.4 System of ODE and boundary value problem

The spectral resolution of $(\frac{1}{2} + \mathcal{O})^{-1}p$ and the interpolation result of Proposition 2.6 reduce boundary value problems for P on $\overline{\mathbb{R}^2_-}$ to an infinite system of ODE's in $\overline{\mathbb{R}_-}$.

2.4.1 Rewriting $Pu = f$

Consider the two cases $I = \mathbb{R}_-$ and $I = \mathbb{R}$. When $u \in L^2(I, dq; \mathcal{H}^1)$ and $f \in L^2(I, dq; \mathcal{H}^{-1})$ the relation $Pu = f$ can be written

$$(\frac{1}{2} + \mathcal{O})^{-1}p\partial_q u + u = \check{f}\,, \tag{23}$$

$$\text{with} \qquad \check{f} = (\frac{1}{2} + \mathcal{O})^{-1} f \in L^2(I, dq; \mathcal{H}^1) = L^2(I, dq; \mathcal{D}_0)\,.$$

Meanwhile the boundary condition, in the case $I = \mathbb{R}_-$, is written for traces in $L^2(\mathbb{R}, |p|dp; \mathfrak{f}) = \mathcal{D}_{-\frac{1}{2}}$. Hence we can use the basis $(e_\nu)_{\nu \in \pm(2\mathbb{N}^*)^{-\frac{1}{2}}}$ and use

$\mathcal{D}_s = \oplus^{\perp}_{\nu\in\pm(2\mathbb{N})^{-\frac{1}{2}}}(\mathbb{C}e_\nu)\otimes\mathfrak{f}$ for all $s\in\mathbb{R}$. By writing

$$u(q,p) = \sum_{\nu\in\pm(2\mathbb{N}^*)^{-\frac{1}{2}}} u_\nu(q)e_\nu\,,\quad u_\nu(q)\in\mathfrak{f}\text{ a.e}$$

the squared $L^2(\mathbb{R}_-,dq;\mathcal{D}_s)$-norm is nothing but

$$\|u\|^2_{L^2(\mathbb{R}_-,dq;\mathcal{D}_s)} = \sum_{\nu\in\pm(2\mathbb{N}^*)^{-\frac{1}{2}}} |\nu|^{2s}\int_{-\infty}^0 |u_\nu(q)|^2_{\mathfrak{f}}\;dq$$

while the trace, when defined in $\mathcal{D}_{-\frac{1}{2}}$ equals $u(0,p) = \sum_{\nu\in\pm(2\mathbb{N}^*)^{-\frac{1}{2}}} u_\nu(0)e_\nu$ with the squared-norm

$$\|u(0,.)\|^2_{\mathcal{D}_{-\frac{1}{2}}} = \sum_{\nu\in\pm(2\mathbb{N}^*)^{-\frac{1}{2}}} |\nu||u_\nu(0)|^2_{\mathfrak{f}}\,.$$

Using the same decomposition for $\check{f}\in L^2(\mathbb{R}_-,dq;\mathcal{D}_0)$, the equation (23) becomes

$$\forall\nu\in\pm(2\mathbb{N}^*)^{-\frac{1}{2}}\,,\quad \nu\partial_q u_\nu + u_\nu = \check{f}_\nu\quad\text{in } I\,.$$

When $I=\mathbb{R}$, the only possible solution is given by

$$\nu>0\,,\qquad u_\nu(q) = \frac{1}{\nu}\int_{-\infty}^q e^{-\frac{q-s}{\nu}}\check{f}_\nu(s)\;ds = (\frac{e^{-\frac{\cdot}{\nu}}}{\nu}1_{\mathbb{R}_+})*\check{f}_\nu(q)$$

$$\nu<0\,,\qquad u_\nu(q) = \frac{1}{\nu}\int_0^q e^{-\frac{(q-s)}{\nu}}\check{f}_\nu(s)\;ds = \left(\frac{1}{\nu}e^{-\frac{\cdot}{\nu}}1_{\mathbb{R}_-}\right)*\check{f}_\nu(q)\,.$$

Actually this formula provides a solution. Owing to the next Lemma.

Lemma 2.8. *For any $s\in\mathbb{R}$ the operator E defined by*

$$E(\sum_{\nu\in\pm(\mathbb{N})^{-\frac{1}{2}}}\check{f}_\nu(q)e_\nu) = \sum_{\nu\in\pm(\mathbb{N})^{-\frac{1}{2}}}\left[\left(\frac{e^{-\frac{\cdot}{\nu}}}{\nu}1_{sign(\nu)\mathbb{R}_+}\right)*\check{f}_\nu\right](q)e_\nu\,,$$

is a contraction on $L^2(\mathbb{R},dq;\mathcal{D}_s)$.

Proof. It suffices to notice that

$$\left\|\left(\frac{e^{-\frac{\cdot}{\nu}}}{\nu}1_{\mathrm{sign}\,(\nu)\mathbb{R}_+}\right)\right\|_{L^1(\mathbb{R},dq)} = 1$$

∎

When $I=\mathbb{R}_-$, we distinguish the two cases $\nu>0$ and $\nu<0$:

$\nu>0$: The solution $u_\nu\in L^2(\mathbb{R}_-,dq;\mathfrak{f})$ to $(\nu\partial_q+1)u_\nu=\check{f}_\nu$ is still

$$u_\nu(q)=\frac{1}{\nu}\int_{-\infty}^{q}e^{-\frac{q-s}{\nu}}\check{f}_\nu(s)\,ds=(\frac{e^{-\frac{\cdot}{\nu}}}{\nu}1_{\mathbb{R}_+})*(\check{f}_\nu 1_{\mathbb{R}_-})(q)\,, \tag{24}$$

which contains

$$u_\nu(0)=\int_{-\infty}^{0}\frac{e^{\frac{s}{\nu}}}{\nu}\check{f}_\nu(s)\,ds\,. \tag{25}$$

$\nu<0$: The possible solutions $u_\nu\in L^2(\mathbb{R}_-,dq;\mathfrak{f})$ to $(\nu\partial_q+1)u_\nu=\check{f}_\nu$ are given by

$$\begin{aligned}u_\nu(q) &= u_\nu(0)e^{-\frac{q}{\nu}}+\frac{1}{\nu}\int_0^q e^{-\frac{(q-s)}{\nu}}\check{f}_\nu(s)\,ds\\ &= u_\nu(0)e^{-\frac{q}{\nu}}+\left(\frac{1}{\nu}e^{-\frac{\cdot}{\nu}}1_{\mathbb{R}_-}\right)*(\check{f}_\nu 1_{\mathbb{R}_-})(q)\,. \end{aligned}\tag{26}$$

2.4.2 Trace theorem and integration by parts

We next prove a trace theorem and an integration by part formula adapted to the Fourier series in $\mathcal{D}_s$.

Definition 2.9. *For an open interval I of $\mathbb{R}$, $I=(a,b)$ with $-\infty\leq a<b\leq+\infty$, one sets*

$$\mathcal{E}_I(\mathfrak{f})=\left\{u\in L^2(I,dq;\mathcal{H}^1)\,,\ p\partial_q u\in L^2(I,dq;\mathcal{H}^{-1})\right\}\,,$$

and its norm is given by

$$\|u\|^2_{\mathcal{E}_I(\mathfrak{f})}=\|u\|^2_{L^2(I,dq;\mathcal{H}^1)}+\|p\partial_q u\|^2_{L^2(I,dq;\mathcal{H}^{-1})}\,.$$

Remember that $\mathcal{H}^s$ is a space of $\mathfrak{f}$-valued distributions.

Proposition 2.10. *The following properties hold:*

- *$\mathcal{E}_I(\mathfrak{f})$ is continuously embedded in $\mathcal{C}^0_b(\overline{I};L^2(\mathbb{R},|p|dp;\mathfrak{f}))=\mathcal{C}^0_b(\overline{I};\mathcal{D}_{-\frac{1}{2}})$;*
- *$\mathcal{S}(\overline{I}\times\mathbb{R};\mathfrak{f})$ is dense in $\mathcal{E}_I(\mathfrak{f})$ and if $a=-\infty$ (resp. $b=+\infty$) the norm $\|u(q,.)\|_{L^2(\mathbb{R},|p|dp;\mathfrak{f})}$ goes to 0 as $q\to-\infty$ (resp. $q\to+\infty$).*

- *Any $u \in \mathcal{E}_I(\mathfrak{f})$ fulfills the integration by parts formula*

$$2\,\mathbb{R}\mathrm{e}\,\langle u\,,\,p\partial_q u\rangle = 1_{\mathbb{R}}(b)\int_{\mathbb{R}}|u(b,p)|_{\mathfrak{f}}^2 pdp - 1_{\mathbb{R}}(a)\int_{\mathbb{R}}|u(a,p)|_{\mathfrak{f}}^2 pdp\,. \quad (27)$$

- *When $a > -\infty$ (resp. $b < +\infty$) the trace map $\gamma_\bullet : \mathcal{E}_I(\mathfrak{f}) \to \gamma_\bullet u = u(\bullet, p) \in L^2(\mathbb{R}, |p|dp; \mathfrak{f}) = \mathcal{D}_{-\frac{1}{2}}$ (with $\bullet = a$ or b respectively) is surjective, with a continuous left-inverse.*

Proof. After noting that $p\partial_q u \in L^2(I, dq; \mathcal{H}^{-1})$ is equivalent to $(\frac{1}{2}+\mathcal{O})^{-1}p\partial_q u \in L^2(I, dq; \mathcal{H}^1)$, introduce the orthonormal basis $(e_\nu)_{\nu\in\pm(2\mathbb{N}^*)^{-\frac{1}{2}}}$. The space $\mathcal{E}_I(\mathfrak{f})$ is the set of series $u = \sum_{\nu\in\pm(2\mathbb{N}^*)^{-\frac{1}{2}}} u_\nu(q)e_\nu$, $u_\nu(q) \in \mathfrak{f}$ a.e., with the norm

$$\|u\|_{\mathcal{E}_I(\mathfrak{f})}^2 = \sum_{\nu\in\pm(2\mathbb{N}^*)^{-\frac{1}{2}}} \int_a^b |u_\nu(q)|_{\mathfrak{f}}^2 + |\nu|^2|\partial_q u_\nu(q)|_{\mathfrak{f}}^2\; dq\,.$$

Every u_ν belongs to $\mathcal{C}_b^0(I;\mathfrak{f})$ and goes to 0 at ∞ when $a = -\infty$ or $b = +\infty$ with

$$\begin{aligned} |\nu||\sup_{q\in I} u_\nu(q)|_{\mathfrak{f}}^2 &\leq C\|u_\nu\|_{L^2(I,dq;\mathfrak{f})}\left[\|u_\nu\|_{L^2(I,dq;\mathfrak{f})} + \|\nu\partial_q u_\nu\|_{L^2(I,dq;\mathfrak{f})}\right] \\ &\leq C'\left[\|u_\nu\|_{L^2(I,dq;\mathfrak{f})}^2 + |\nu|^2\|\partial_q u_\nu\|_{L^2(I,dq;\mathfrak{f})}^2\right] \leq C'\|u\|_{\mathcal{E}_I(\mathfrak{f})}^2\,. \end{aligned}$$

The dominated convergence theorem applied to the series

$$\sum_{\nu\in\pm(2\mathbb{N}^*)^{-\frac{1}{2}}} |\nu|\,|u_\nu(q) - u_\nu(q_0)|_{\mathfrak{f}}^2 \quad \text{as} \quad q \to q_0\,,$$

with the upper bound

$$|\nu||u_\nu(q) - u_\nu(q_0)|_{\mathfrak{f}}^2 \leq 2C'\left[\|u_\nu\|_{L^2(I,dq;\mathfrak{f})}^2 + |\nu|^2\|\partial_q u_\nu\|_{L^2(I,dq;\mathfrak{f})}^2\right]\,,$$

provides the $\mathcal{D}_{-\frac{1}{2}}$-continuity w.r.t q.
For the density of $\mathcal{S}(\overline{I}\times\mathbb{R};\mathfrak{f})$, it suffices to approximate $u = \sum_{\nu\in\pm(2\mathbb{N}^*)^{-\frac{1}{2}}} u_\nu(q)e_\nu$ by a finite sum $u^N = \sum_{|\nu|\geq N^{-1}} u_\nu^N(q)e_\nu$, with $u_\nu^N \in \mathcal{S}(\overline{I};\mathfrak{f})$,

$$\|u_\nu^N\|_{L^2(I,dq;\mathfrak{f})}^2 + |\nu|^2\|\partial_q u_\nu^N\|_{L^2(I,dq;\mathfrak{f})}^2 \leq 2\|u_\nu\|_{L^2(I,dq;\mathfrak{f})}^2 + 2|\nu|^2\|\partial_q u_\nu\|_{L^2(I,dq;\mathfrak{f})}^2\,,$$

and $\lim_{N\to\infty} \|u_\nu^N - u_\nu\|_{L^2(I,dq;\mathfrak{f})}^2 + |\nu|^2\|\partial_q(u_\nu^N - u_\nu)\|_{L^2(I,dq;\mathfrak{f})}^2 = 0$

for all $\nu \in \pm(2\mathbb{N}^*)^{-\frac{1}{2}}$.
The integration by parts formula (27) is true when $u \in \mathcal{S}(\overline{I} \times \mathbb{R}; \mathfrak{f})$ and all its terms are continuous on $\mathcal{E}_I(\mathfrak{f})$.
For the surjectivity, the translation invariance allows to assume $b = 0$. Take $\gamma = \sum_{\nu \in (2\mathbb{N}^*)^{-1/2}} \gamma_\nu e_\nu$ with $\|\gamma\|^2_{\mathcal{D}_{-\frac{1}{2}}} = \sum_{\nu \in (2\mathbb{N}^*)^{-1/2}} |\nu||\gamma_\nu|^2_{\mathfrak{f}} < \infty$. Choose a non negative cut-off function $\chi \in \mathcal{C}_0^\infty((a, 0])$ such that $\chi \equiv 1$ in the neighborhood of 0. We use the presentation of Subsection 2.4.1 and write $Pu = f$ in the form $((\frac{1}{2} + \mathcal{O})^{-1} p\partial_q + 1)u = \check{f} = (\frac{1}{2} + \mathcal{O})^{-1} f$.
For $\nu < 0$, take

$$u_\nu = \gamma_\nu \chi(\frac{q}{\nu}) \quad \text{and} \quad \check{f}_\nu = (\nu\partial_q u_\nu + u_\nu) = \gamma_\nu(\chi'(\frac{q}{\nu}) + \chi(\frac{q}{\nu}))\,.$$

For $\nu > 0$, take $\check{f}_\nu(q) = 2\gamma_\nu \chi(\frac{q}{\nu})$ and

$$\begin{aligned} u_\nu(q) &= \frac{\gamma_\nu}{\nu \int_{-\infty}^0 e^s \chi(s)\ ds} \int_{-\infty}^q e^{-\frac{q-s}{\nu}} \chi(\frac{s}{\nu})\ ds \\ &= \frac{1}{\int_{-\infty}^0 e^s \chi(s)\ ds} \left(\frac{e^{-\frac{\cdot}{\nu}}}{\nu} 1_{\mathbb{R}_+}\right) * (\chi(\frac{\cdot}{\nu}) 1_{\mathbb{R}_-})\gamma_\nu\,. \end{aligned}$$

With $\left\|\left(\frac{e^{-\frac{\cdot}{\nu}}}{\nu} 1_{\mathbb{R}_+}\right)\right\|_{L^1} = 1$ and $\|\gamma_\nu \phi(\frac{\cdot}{\nu})\|^2_{L^2(\mathbb{R}_-, dq;\mathfrak{f})} = |\nu||\gamma_\nu|^2_{\mathfrak{f}} \int_{-\infty}^0 |\phi(s)|^2\ ds$ with $\phi = \chi$ or $\phi = \chi' + \chi$, we deduce $u, \check{f} \in L^2(\mathbb{R}_-, dq; \mathcal{H}^1)$ and therefore $u \in L^2(\mathbb{R}_-, dq; \mathcal{H}^1)$ and $Pu = f = (\frac{1}{2} + \mathcal{O})\check{f} \in L^2(\mathbb{R}_-, dq; \mathcal{H}^{-1})$. Meanwhile the integral $\int_{-\infty}^0 e^{\frac{s}{\nu}} \chi(s)\ ds = \nu \int_{-\infty}^0 e^s \chi(s)\ ds$ for $\nu > 0$, leads to $u(q = 0) = \gamma$. Our construction also ensures $\operatorname{supp} u \subset \{q \in \operatorname{supp} \chi\} \subset \{q \in (a, b = 0]\}$.
The surjectivity for the trace at a when $a > -\infty$ is deduced by the symmetry $(q, p) \to (a + b - q, -p)$ when $-\infty < a < b < +\infty$ or $(q, p) \to (2a - q, -p)$ when $-\infty < a$, $b = +\infty$. □

2.4.3 Calderon projector and boundary value problem

The operator $\tilde{P} = \frac{1}{2} + P$ is a local (differential operator) on $\mathbb{R}^2$ and we construct the associated Calderon projector at $q = 0$ for boundary value problems on $\overline{\mathbb{R}^2_-}$. For Cauchy problems for ODE's with constant coefficients, the Calderon projector selects the relevant exponentially decaying modes. We recall and apply the general definition for differential operators. Let us first check the invertibility of $\tilde{P} = \frac{1}{2} + P$ on the whole space $\mathbb{R}^2$.

Proposition 2.11. *The operator* $\tilde{P} = \frac{1}{2} + P = p\partial_q + (\frac{1}{2} + \mathcal{O})$ *defines an isomorphism from* $\mathcal{E}_{\mathbb{R}}(\mathfrak{f})$ *to* $L^2(\mathbb{R}, dq; \mathcal{H}^{-1})$.

Proof. When $u \in \mathcal{E}_{\mathbb{R}}(\mathfrak{f})$, $(\frac{1}{2} + \mathcal{O})u \in L^2(\mathbb{R}, dq; \mathcal{H}^{-1})$ and

$$\begin{aligned}\|\tilde{P}u\|_{L^2(\mathbb{R},dq;\mathcal{H}^{-1})} &= \|p\partial_q u + (\frac{1}{2} + \mathcal{O})u\|_{L^2(\mathbb{R},dq;\mathcal{H}^{-1})} \\ &\leq \|p\partial_q u\|_{L^2(\mathbb{R},dq;\mathcal{H}^{-1})} + \|(\frac{1}{2} + \mathcal{O})u\|_{L^2(\mathbb{R},dq;\mathcal{H}^{-1})} \leq 2\|u\|_{\mathcal{E}_{\mathbb{R}}(\mathfrak{f})} .\end{aligned}$$

When $f \in L^2(\mathbb{R}, dq; \mathcal{H}^{-1} \otimes \mathfrak{f})$, the rewriting of $\tilde{P}u = f$ as $(\frac{1}{2} + \mathcal{O})^{-1} p\partial_q u + u = \check{f} = (\frac{1}{2} + \mathcal{O})^{-1} f \in L^2(\mathbb{R}, dq; \mathcal{H}^{-1} \otimes \mathfrak{f})$ like in Subsection 2.4.1, combined with Lemma 2.8 (case $s = 0$) , provides a solution $u \in L^2(\mathbb{R}, dq; \mathcal{H}^1 \otimes \mathfrak{f})$ to $\tilde{P}u = f$. If there are two solutions $u_1, u_2 \in \mathcal{E}_{\mathbb{R}}(\mathfrak{f})$ to $\tilde{P}u = f$, the difference $u_2 - u_1$ belongs to $\mathcal{E}_{\mathbb{R}}(\mathfrak{f})$ and solves $p\partial_q(u_2 - u_1) = -(\frac{1}{2} + \mathcal{O})(u_2 - u_1)$. The integration by parts (27) with $a = -\infty$ and $b = +\infty$ gives

$$-\operatorname{Re}\langle (u_2 - u_1), (\frac{1}{2} + \mathcal{O})(u_2 - u_1)\rangle = \operatorname{Re}\langle (u_2 - u_1), p\partial_q(u_2 - u_1)\rangle = 0 .$$

This means $\|u_2 - u_1\|_{L^2(\mathbb{R},dq;\mathcal{H}^1)} = 0$ and $u_2 = u_1$. ☐

Definition 2.12. *When* $\mathfrak{g}$ *is a separable Hilbert space, we call* r_- *the restriction operator*

$$r_- : L^2(\mathbb{R}, dq; \mathfrak{g}) \to L^2(\mathbb{R}_-, dq; \mathfrak{g}) \quad r_- u = u\big|_{\mathbb{R}_-} ,$$

and e_- *the extension by* 0 *,* $e_- = r_-^*$ *,*

$$e_- : L^2(\mathbb{R}_-, dq; \mathfrak{g}) \to L^2(\mathbb{R}, dq; \mathfrak{g}) \quad , \quad e_- u = u \times 1_{\mathbb{R}_-} .$$

Proposition 2.13. *For* $\tilde{P} = \frac{1}{2} + P$ *, the expression*

$$\mathcal{K}\gamma = u - r_- \tilde{P}^{-1} e_- \tilde{P} u \quad \textit{for } \gamma_0 u = \gamma ,$$

with $u \in \mathcal{E}_{\mathbb{R}_-}(\mathfrak{f})$ *defines a continuous operator from* $L^2(\mathbb{R}, |p|dp; \mathfrak{f})$ *to* $\mathcal{E}_{\mathbb{R}_-}(\mathfrak{f})$ *such that* $\tilde{P}(\mathcal{K}\gamma) = 0$ *for all* $\gamma \in L^2(\mathbb{R}, |p|dp; \mathfrak{f})$ *.*
The operator $C_0 = \gamma_0 \circ \mathcal{K}$ *is the orthogonal projection* $\frac{1-S}{2} = 1_{\mathbb{R}_-}(S)$ *on* $L^2(\mathbb{R}, |p|dp; \mathfrak{f}) = \mathcal{D}_{-\frac{1}{2}}$ *.*

Proof. We know that $\gamma_0 : u \in \mathcal{E}_{\mathbb{R}_-}(\mathfrak{f}) \to \gamma_0 u \in L^2(\mathbb{R}, |p|dp; \mathfrak{f})$ is surjective.
For every $u \in \mathcal{E}_{\mathbb{R}_-}(\mathfrak{f})$, the continuity of $\tilde{P} : \mathcal{E}_{\mathbb{R}_-}(\mathfrak{f}) \to L^2(\mathbb{R}_-, dq; \mathcal{H}^{-1})$ and Proposition 2.11 ensure that

$$v_u = u - r_- \tilde{P}^{-1} e_- \tilde{P} u\,,$$

belongs to $\mathcal{E}_{\mathbb{R}_-}(\mathfrak{f})$ and solves $\tilde{P} v_u = 0$ for $q < 0$.
For $\gamma \in L^2(\mathbb{R}, |p|dp; \mathfrak{f})$ assume that $u_1 \in \mathcal{E}_{\mathbb{R}_-}(\mathfrak{f})$ and $u_2 \in \mathcal{E}_{\mathbb{R}_-}(\mathfrak{f})$ satisfy $\gamma_0 u_2 = \gamma_0 u_1 = \gamma$. The difference $u_2 - u_1$ has a null trace at $q = 0$ and since $\tilde{P}$ is a first order differential operator in q, one gets the equality $e_- P u = P(e_- u)$. The invertibility of $\tilde{P} : \mathcal{E}_{\mathbb{R}}(\mathfrak{f}) \to L^2(\mathbb{R}_-, dq; \mathcal{H}^{-1})$ (already used to define v_u) provides the nullity

$$v_{u_2} - v_{u_1} = u_2 - u_1 - r_- \tilde{P}^{-1} \tilde{P} e_-(u_2 - u) = 0 \quad \text{for } q < 0\,.$$

The possibility to choose $u \in \mathcal{E}_{\mathbb{R}_-}(\mathfrak{f})$ such that $\gamma_0 u = \gamma$ and $\|u\|_{\mathcal{E}_{\mathbb{R}_-}(\mathfrak{f})} \leq C\|\gamma\|_{L^2(\mathbb{R},|p|dp;\mathfrak{f})}$ according to Proposition 2.10, combined with $\|v_u\|_{\mathcal{E}_{\mathbb{R}_-}(\mathfrak{f})} \leq C'\|u\|_{\mathcal{E}_{\mathbb{R}_-}(\mathfrak{f})}$, implies the continuity of $\mathcal{K}$.
With the rewriting of $\tilde{P}u = f$ as $(\frac{1}{2} + \mathcal{O})^{-1}\tilde{P}u = \check{f} = (\frac{1}{2} + \mathcal{O})^{-1} f$, $\mathcal{K}\gamma$ equals

$$\mathcal{K}\gamma = u - r_- \tilde{P}^{-1} e_- \tilde{P} u = u - r_- \tilde{P}^{-1}(\frac{1}{2} + \mathcal{O}) e_-(\frac{1}{2} + \mathcal{O})^{-1} \tilde{P} u\,.$$

After introducing the orthogonal decomposition $((\mathbb{C}e_\nu) \otimes \mathfrak{f})_{\nu \in \pm(2\mathbb{N}^*)^{-\frac{1}{2}}}$ of $\mathcal{H}^1$ and $\mathcal{D}_{-\frac{1}{2}} = L^2(\mathbb{R}, |p|dp)$, it becomes

$$(\mathcal{K}\gamma)_\nu(q) = u_\nu(q) - \left[r_-(\nu\partial_q + 1)^{-1} e_-(\nu\partial_q + 1) u_\nu\right](q)\,, \quad \forall \nu \in \pm(2\mathbb{N}^*)^{-\frac{1}{2}}\,.$$

All the u_ν's belong to $H^1_{loc}(\mathbb{R}_-)$ and the equality

$$e_-(\nu\partial_q + 1) u_\nu = (\nu\partial_q + 1) e_- u_\nu + \nu u_\nu(0)\delta_0\,,$$

holds in the distributional sense in $\mathbb{R}$. The general solution on $\mathbb{R}$ to $(\nu\partial_q + 1)v = \nu u_\nu(0)\delta_0$ in $\mathbb{R}$ is

$$1_{\mathbb{R}_-}(q) e^{-\frac{q}{\nu}} C + (u_\nu(0) + C) 1_{\mathbb{R}_+}(q) e^{-\frac{q}{\nu}}\,, \quad C \in \mathfrak{f}\,.$$

Thus, the only possible value for $(\nu\partial_q + 1)^{-1} e_-(\nu\partial_q + 1) u_\nu$ is given by

$$\begin{aligned} &u_\nu(q) 1_{\mathbb{R}_-}(q) + u_\nu(0) e^{-\frac{q}{\nu}} 1_{\mathbb{R}_+}(q)\,, \quad \text{for } \nu > 0\,, \\ &u_\nu(q) 1_{\mathbb{R}_-}(q) - u_\nu(0) 1_{\mathbb{R}_-}(q) e^{-\frac{q}{\nu}}\,, \quad \text{for } \nu < 0\,. \end{aligned}$$

We finally obtain after taking the restriction to $\mathbb{R}_-$,

$$(K\gamma)_\nu(q) = \begin{cases} 0 & \text{for } \nu > 0\,, \\ u_\nu(0)1_{\mathbb{R}_-}(q)e^{-\frac{q}{\nu}} = \gamma_\nu 1_{\mathbb{R}_-}(q)e^{-\frac{q}{\nu}} & \text{for } \nu < 0\,. \end{cases}$$

Taking the trace at $q = 0$ yields

$$(C_0\gamma)_\nu = \gamma_\nu 1_{\mathbb{R}_-}(\nu) = \left(1_{\mathbb{R}_-}(S)\gamma\right)_\nu\,, \quad \forall \nu \in \pm(2\mathbb{N}^*)^{-\frac{1}{2}}\,.$$

$\square$

Remark 2.14. *In general the local nature of $\tilde{P}$ suffices to prove that the Calderon projector C_0 fulfills $C_0 \circ C_0 = C_0$. Here (actually all the construction was made for this) we identify directly $C_0 = 1_{\mathbb{R}_-}(S) = \frac{1-S}{2}$. The range of C_0 is nothing but the kernel of $1_{\mathbb{R}_+}(S) = \frac{1+S}{2}$.*
The operator $\mathcal{K}$ is usually called a Poisson operator.

The Calderon projector is used to study the well-posedness of boundary value problems (see [ChPi][BdM][HormIII]-Chap 20 or [Tay]). This is summarized in the following straightforward proposition.

Proposition 2.15. *Set $\tilde{P} = \frac{1}{2} + P$. Let $\mathcal{T}$ be a Hilbert spaces and let $T : \mathcal{D}_{-\frac{1}{2}} \to \mathcal{T}$ be a continuous operator. Consider the boundary value problem*

$$\begin{cases} \tilde{P}u = f \in L^2(\mathbb{R}_-, dq; \mathcal{H}^{-1}) \\ T\gamma_0 u = f_\partial \in \mathcal{T}\,, \end{cases} \qquad u \in \mathcal{E}_{\mathbb{R}_-}(\mathfrak{f})\,. \tag{28}$$

Call T_- the restriction $T\big|_{Ran\, C_0} = T\big|_{\ker(1+S)}$. When T_- is injective the boundary value problem (28) admits at most one solution in $\mathcal{E}_{\mathbb{R}}(\mathfrak{f})$. When T_- is surjective the boundary value problem (28) has solutions for any $f_\partial \in \mathcal{T}$ and $f \in L^2(\mathbb{R}, dq; \mathcal{H}^{-1})$. When T_- is an isomorphism, then (28) admits a unique solution

$$u = r_-\tilde{P}^{-1}e_- f + \mathcal{K}T_-^{-1}[f_\partial - T\gamma_0(r_-\tilde{P}^{-1}e_- f)]\,.$$
$$\textit{with} \qquad \|u\| \leq C\left[\|f\|_{L^2(\mathbb{R}_-,\mathcal{H}^{-1})} + \|f_\partial\|_{\mathcal{T}}\right]\,.$$

Since $\mathrm{Ran}\, C_0$ is the kernel of the simple operator $\frac{(1+S)}{2} = 1_{\mathbb{R}_+}(S)$, one can study the well posedness of the boundary value problem by considering $T : \mathcal{D}_{-\frac{1}{2}} = L^2(\mathbb{R}, |p|dp; \mathfrak{f}) \to \mathcal{T}$ and by studying whether the system

$$\begin{cases} T\gamma = f'_\partial \in \mathcal{T} \\ (1+S)\gamma = 0 \end{cases} \tag{29}$$

admits a unique solution $\gamma \in L^2(\mathbb{R}, |p|dp; \mathfrak{f}) = \mathcal{D}_{-\frac{1}{2}}$ for any $f'_\partial \in \mathcal{T}$. A typical example is $\mathcal{T} = \operatorname{Ran} C_0$ and $T = C_0 = \frac{1-S}{2} = 1_{\mathbb{R}_-}(S)$. The boundary value problem

$$\begin{cases} \tilde{P}u = f\,, \\ 1_{\mathbb{R}_-}(S)\gamma_0 u = f_\partial\,, \end{cases}$$

is well-posed in $\mathcal{E}_{\mathbb{R}_-}(\mathfrak{f})$ for all $f \in L^2(\mathbb{R}_-, dq; \mathcal{H}^{-1})$ and all $f_\partial \in \operatorname{Ran} C_0 = \operatorname{Ran} 1_{\mathbb{R}_-}(S)$.
A solution $u \in \mathcal{E}_{\mathbb{R}_-}(\mathfrak{f})$ to $\tilde{P}u = f \in L^2(\mathbb{R}_-, dq; \mathcal{H}^{-1})$ coincides with the restriction to $\mathbb{R}_-$ of $\tilde{P}^{-1}e_- f = \tilde{P}^{-1}(f 1_{\mathbb{R}_-}(q))$ if and only if $C_0 u = 0$, that is $1_{\mathbb{R}_-}(S)u = 0$.

2.5 Maximal accretivity

In order to prove Theorem 2.1, we study the boundary value problem

$$Pu = f \quad , \quad \gamma_{odd} u = \operatorname{sign}(p) \times A\gamma_{ev} u\,,$$

in a larger framework by considering $f \in L^2(\mathbb{R}_-, dq; \mathcal{H}^{-1})$ and $u \in L^2(\mathbb{R}_-, dq; \mathcal{H}^1)$ better suited for functional analysis arguments.
Remember $\gamma u = \gamma_0 u = u\big|_{q=0}$ and $\gamma_{ev,odd} u = \Pi_{ev,odd}\gamma u$.

2.5.1 Boundary value problem related with A

Although they are equal, the Hilbert spaces $L^2(\mathbb{R}, |p|dp, \mathfrak{f})$ and $\mathcal{D}_{-\frac{1}{2}}$ are endowed with two different scalar products. When $(A, D(A))$ is maximal accretive in $L^2(\mathbb{R}, |p|dp; \mathfrak{f})$ commuting with Π_{ev}, the same holds for $(MA, D(MA))$ in $D_{-\frac{1}{2}}$, with $M = S \circ \operatorname{sign}(p)$ and $D(MA) = D(A)$, by Lemma 2.7 and the commutation $M\Pi_{ev} = \Pi_{ev} M$. We shall first work in $\mathcal{D}_{-\frac{1}{2}}$. For the corresponding scalar product, the adjoint $((MA)^*, D((MA)^*))$ is also maximal accretive owing to Hille-Yosida theorem and $\|(\lambda + (MA)^*)^{-1}\|_{\mathcal{L}(\mathcal{D}_{-\frac{1}{2}})} = \|(\lambda + MA)^{-1}\|_{\mathcal{L}(\mathcal{D}_{-\frac{1}{2}})} \leq \frac{1}{\lambda}$ for $\lambda > 0$. The operator $(1 + MA)$ (resp. $1 + MA^*$) defines isomorphisms in the diagram

$$\begin{array}{rl} & D(A) \overset{1+MA}{\to} \mathcal{D}_{-\frac{1}{2}} \overset{1+MA}{\to} D((MA)^*)' \\ \text{resp.} & D((MA)^*) \overset{1+(MA)^*}{\to} \mathcal{D}_{-\frac{1}{2}} \overset{1+(MA)^*}{\to} D(MA)'\,, \end{array}$$

where $D((MA)^*)'$ (resp. $D(MA)'$) is the dual of $D((MA)^*)$ (resp. $D(MA)$). This will be more systematically used in Section 4. Those isomorphisms and the commutation $\Pi_{ev}MA = MA\Pi_{ev}$ ensure $E = \Pi_{ev}E \oplus^{\perp} \Pi_{odd}E$ for $E = D(MA), D((MA)^*),\ D(MA)', D((MA)^*)'$, endowed with the suitable scalar product.

Proposition 2.16. *Keep the notation $\tilde{P} = \frac{1}{2} + P$. Then for any $f \in L^2(\mathbb{R}_-, dq; \mathcal{H}^{-1})$ and any $f_\partial \in \Pi_{ev}D((MA)^*)'$, the boundary value problem*

$$\tilde{P}u = f \quad , \quad S\gamma_{odd}u - MA\gamma_{ev}u = f_\partial\,, \tag{30}$$

admits a unique solution in $u \in L^2(\mathbb{R}_-, dq; \mathcal{H}^1)$, which belongs to $\mathcal{E}_{\mathbb{R}_-}(\mathfrak{f}) \subset \mathcal{C}^0_b((-\infty, 0]; L^2(\mathbb{R}, |p|dp; \mathfrak{f})$ and satisfies

$$\|u\|_{\mathcal{E}_{\mathbb{R}_-}(\mathfrak{f})} \leq C\left[\|f\|_{L^2(\mathbb{R}_-,dq;\mathcal{H}^{-1})} + \|(1+MA)^{-1}f_\partial\|_{\mathcal{D}_{-\frac{1}{2}}}\right].$$

Proof. The conditions $u \in L^2(\mathbb{R}_-, dq; \mathcal{H}^1)$ and $\tilde{P}u \in L^2(\mathbb{R}_-, dq; \mathcal{H}^{-1})$ imply $u \in \mathcal{E}_{\mathbb{R}_-}(\mathfrak{f})$ and $\gamma u \in \mathcal{D}_{-\frac{1}{2}}$. Since $S\Pi_{odd}\gamma u = \Pi_{ev}S\gamma u$ and $MA\Pi_{ev}\gamma u \in \Pi_{ev}D((MA)^*)'$, the boundary condition makes sense for $f_\partial \in \Pi_{ev}D((MA)^*)'$. According to Proposition 2.15 and the reformulation (29), the boundary value problem (30) is well posed if the system

$$\left\{\begin{array}{l} S\gamma_{odd} - MA\gamma_{ev} = f'_\partial \ , \quad \text{with } \gamma_{ev,odd} = \Pi_{ev,odd}\gamma\,,\\ (1+S)\gamma = 0 \end{array}\right.$$

admits a unique solution $\gamma \in \mathcal{D}_{-\frac{1}{2}}$ for all $f'_\partial \in \Pi_{ev}D((MA)^*)'$. After (18)(19) we checked $S\Pi_{ev} = \Pi_{odd}S$ and the above system is equivalent to

$$\left\{\begin{array}{l} S\gamma_{odd} - MA\gamma_{ev} = f'_\partial\,,\\ \Pi_{ev}(1+S)\gamma = 0\,,\\ \Pi_{odd}(1+S)\gamma = 0 \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} (\mathrm{Id} + MA)\gamma_{ev} = -f'_\partial\,,\\ \gamma_{odd} = -S\gamma_{ev}\,, \end{array}\right.$$

which admits a unique solution $\gamma \in \mathcal{D}_{-\frac{1}{2}}$ such that $\|\gamma\|_{\mathcal{D}_{-\frac{1}{2}}} \leq C_1\|f'_\partial\|_{D((MA)^*)'}$. The final estimate is also provided by Proposition 2.15. $\square$

We now translate the previous result in the more usual structure on $L^2(\mathbb{R}, |p|dp; \mathfrak{f})$. By making use of the integration by part formula (27), the estimate of $\|u\|_{L^2(\mathbb{R}_-,dq;\mathcal{H}^1)}$ will also be made more accurate.

Proposition 2.17. *Set $\tilde{P} = (\frac{1}{2} + P)$ and assume $(A, D(A))$ maximal accretive in $L^2(\mathbb{R}, |p|dp; \mathfrak{f})$ and commuting with Π_{ev} . For any $f \in L^2(\mathbb{R}_-, dq; \mathcal{H}^{-1})$ and $f_\partial \in \Pi_{odd} L^2(\mathbb{R}, |p|dp; \mathfrak{f})$ the boundary value problem*

$$\tilde{P}u = f \quad , \quad \gamma_{odd}u - sign\,(p) A\gamma_{ev}u = f_\partial \tag{31}$$

admits a unique solution $u \in L^2(\mathbb{R}_-, dq; \mathcal{H}^1)$ which belongs to $\mathcal{E}_{\mathbb{R}_-}(\mathfrak{f}) \subset \mathcal{C}_b^0((-\infty, 0]; L^2(\mathbb{R}, |p|dp; \mathfrak{f})$ with $\gamma_{ev}u \in D(A)$. This solution satisfies

$$\begin{aligned}\mathbb{R}\mathrm{e}\,\langle \gamma_{ev}u\,,\, A\gamma_{ev}u\rangle_{L^2(\mathbb{R},|p|dp;\mathfrak{f})} + \|u\|^2_{L^2(\mathbb{R}_-,dq;\mathcal{H}^1)} &= \mathbb{R}\mathrm{e}\,\langle f\,,\, u\rangle \\ &\quad - \mathbb{R}\mathrm{e}\int_{\mathbb{R}} \langle f_\partial(p), \gamma_{ev}u(p)\rangle_{\mathfrak{f}}\; pdp\,.\end{aligned}$$

When $f_\partial = 0$ this implies

$$\|u\|_{L^2(\mathbb{R}_-,dq;\mathcal{H}^1)} \leq \|f\|_{L^2(\mathbb{R}_-,dq;\mathcal{H}^{-1})}\,.$$

Proof. The boundary value problem (30) with $f_\partial \in \Pi_{ev}\mathcal{D}_{-\frac{1}{2}}$ admits a unique solution such that $\gamma_{ev}u \in \mathcal{D}_{-\frac{1}{2}}$ and

$$MA\gamma_{ev}u = f_\partial - S\gamma_{odd}u \in \mathcal{D}_{-\frac{1}{2}}\,.$$

Therefore $\gamma_{ev}u \in D(MA) = D(A)$ and, with $M = S \circ \mathrm{sign}\,(p)$, the boundary value problem (30) is equivalent to

$$\tilde{P}u = f \quad , \quad \gamma_{odd}u - \mathrm{sign}\,(p) A\gamma_{ev}u = Sf_\partial\,,$$

which is (31) after changing Sf_∂ into f_∂ .
We can now use the integration by parts formula (27) which gives

$$\begin{aligned}0 \leq \mathbb{R}\mathrm{e}\,\langle \gamma_{ev}u\,,\, A\gamma_{ev}u\rangle_{L^2(\mathbb{R},|p|dp;\mathfrak{f})} &= +\,\mathbb{R}\mathrm{e}\,\langle \gamma_{ev}u\,,\, \mathrm{sign}\,(p)\gamma_{odd}u\rangle_{L^2(\mathbb{R},|p|dp;\mathfrak{f})} \\ &\quad - \mathbb{R}\mathrm{e}\,\langle \gamma_{ev}u\,,\, \mathrm{sign}\,(p) f_\partial\rangle_{L^2(\mathbb{R},|p|dp;\mathfrak{f})} \\ &= \frac{1}{2}\int_{\mathbb{R}} |u(0,p)|^2_{\mathfrak{f}}\; pdp - \mathbb{R}\mathrm{e}\int_{\mathbb{R}} \langle f_\partial(p)\,,\, \gamma_{ev}u(p)\rangle_{\mathfrak{f}}\; pdp \\ &= \mathbb{R}\mathrm{e}\,\langle u\,, p\partial_q u\rangle - \mathbb{R}\mathrm{e}\int_{\mathbb{R}} \langle f_\partial(p)\,,\, \gamma_{ev}u(p)\rangle_{\mathfrak{f}}\; pdp \\ &= \mathbb{R}\mathrm{e}\,\langle u\,, Pu\rangle - \|u\|^2_{L^2(\mathbb{R},dp;\mathcal{H}^1)} - \mathbb{R}\mathrm{e}\int_{\mathbb{R}} \overline{f_\partial(p)}\gamma_{ev}u(p)\; pdp\,.\end{aligned}$$

$\square$

Remark 2.18. *Note that only the even part $A_{ev} = \Pi_{ev} A \Pi_{ev}$ appears in the assumption and all the analysis. So only this restriction can be considered with an arbitrary maximal accretive extension on* $\mathrm{Ran}\,\Pi_{odd}$ *. In applications, it is easier to consider a realization of $(A, D(A))$ in $L^2(\mathbb{R}, |p|dp; \mathfrak{f})$ without the parity in the domain definition and just check $A\Pi_{ev} = \Pi_{ev} A$.*

2.5.2 Maximal accretivity of K_A

We end here the proof of Theorem 2.1.

Proof. The domain $D(K_A)$ contains $\mathcal{C}_0^\infty((-\infty, 0) \times (\mathbb{R} \setminus \{0\}); \mathfrak{f})$. It is dense in $L^2(\mathbb{R}_-^2, dqdp; \mathfrak{f})$.
From Proposition 2.17 applied with $f_\partial = 0$, we know that $\|u\|_{E_A} = \|(\frac{1}{2} + P)u\|_{L^2(\mathbb{R}_-, dq; \mathcal{H}^{-1})}$ is a norm on

$$E_A = \left\{u \in L^2(\mathbb{R}_-, dq; \mathcal{H}^1)\,, Pu \in L^2(\mathbb{R}_-, dq; \mathcal{H}^{-1})\,, \gamma_{odd} u = \mathrm{sign}\,(p) A \gamma_{ev} u\right\}\,,$$

which is made a Banach space isomorphic via $(\frac{1}{2} + P)$ to $L^2(\mathbb{R}, dq; \mathcal{H}^{-1})$ and continuously embedded in

$$\begin{aligned} & \mathcal{E}_{\mathbb{R}_-}(\mathfrak{f}) \subset L^2(\mathbb{R}_-, dq; \mathcal{H}^1) \cap \mathcal{C}_b^0((-\infty, 0]; L^2(\mathbb{R}, |p|dp; \mathfrak{f}) \\ \text{with} \quad & (u \in E_A) \Rightarrow (\gamma_{ev} u \in D(A))\,. \end{aligned}$$

Since $L^2(\mathbb{R}_-^2, dqdp; \mathfrak{f})$ is complete and continuously embedded in $L^2(\mathbb{R}_-, dq; \mathcal{H}^{-1})$, the domain $D(K_A)$ endowed with the graph norm $\|u\|_{D(K_A)} = \|u\| + \|(\frac{1}{2} + P)u\|$ is complete, which means that K_A is closed.
By Proposition 2.17 we also know that $\frac{1}{2} + P : E_A \to L^2(\mathbb{R}, dq; \mathcal{H}^{-1})$ is surjective, which implies $\mathrm{Ran}\, K_A = L^2(\mathbb{R}_-^2, dqdp; \mathfrak{f})$.
For the accretivity and the integration by part formula (16), simply use Proposition 2.17 with $f_\partial = 0$:

$$\mathrm{Re}\,\langle u\,,\,(K_A - \frac{1}{2})u\rangle = \mathrm{Re}\,\langle \gamma_{ev} u\,,\, A\gamma_{ev} u\rangle_{L^2(\mathbb{R}, |p|dp; \mathfrak{f})} + \|u\|^2_{L^2(\mathbb{R}_-, dq; \mathcal{H}^1)} - \|u\|^2 \geq 0\,.$$

□

2.6 Extension of the resolvent and adjoint

We end the proofs of Proposition 2.2 and Proposition 2.3.

Proof of Proposition 2.2. For $f \in L^2(\mathbb{R}_-, dq; \mathcal{H}^1)$ and $z \in \mathbb{C}$ a solution $u \in L^2(\mathbb{R}_-, dq; \mathcal{H}^1)$

$$(P - z)u = f \quad , \quad \gamma_{odd} f = \mathrm{sign}\,(p) A \gamma_{ev} u$$

satisfies $p\partial_q u = f + zu - \mathcal{O}u \in L^2(\mathbb{R}_-, dq; \mathcal{H}^{-1})$. It necessarily belongs to $\mathcal{E}_{\mathbb{R}_-}(\mathfrak{f})$ and satisfies $\gamma_{ev} u \in D(A)$.
Uniqueness: If there are two solutions u_1, u_2, the difference $u = u_2 - u_1$, solves $(P - z)u = 0 \in L^2(\mathbb{R}^2_-, dqdp; \mathfrak{f})$. When $\mathbb{R}\mathrm{e}\ z < \frac{1}{2}$, it means $u = (K_A - z)^{-1} 0 = 0$.
Existence: From the integration by part of Theorem 2.1 an element $u \in D(K_A)$ satisfies

$$\begin{aligned}\mathbb{R}\mathrm{e}\,\langle u\,,\,(K_A - z)u\rangle &\geq \|u\|^2_{L^2(\mathbb{R}_-, dq; \mathcal{H}^1)} - (\frac{1}{2} + \mathbb{R}\mathrm{e}\ z)\|u\|^2 \\ &\geq (\frac{1}{2} - \mathbb{R}\mathrm{e}\ z)\|u\|^2\,,\end{aligned}$$

and for $\mathbb{R}\mathrm{e}\ z < \frac{1}{2}$

$$\left(1 + \frac{1}{\frac{1}{2} - \mathbb{R}\mathrm{e}\ z}\right) \|u\|_{L^2(\mathbb{R}_-, dq; \mathcal{H}^1)} \|(K_A - z)u\|_{L^2(\mathbb{R}_-, dq; \mathcal{H}^{-1})} \geq \|u\|^2_{L^2(\mathbb{R}_-, dq; \mathcal{H}^1)}\,.$$

Hence $(K_A - z)^{-1}$ has a unique continuous extension from $L^2(\mathbb{R}_-, dq; \mathcal{H}^{-1})$ to $L^2(\mathbb{R}_-, dq; \mathcal{H}^1)$.
For $f \in L^2(\mathbb{R}_-, dq; \mathcal{H}^{-1})$ and $\mathbb{R}\mathrm{e}\ z < \frac{1}{2}$, the function $u = (K_A - z)^{-1} f$ satisfies $(P - z)u = f$ in $\mathcal{D}'(\mathbb{R}^2_-; \mathfrak{f})$. It belongs to $\mathcal{E}_{\mathbb{R}_-}(\mathfrak{f})$ and the traces $\gamma_{ev,odd} u$ are well defined. When $f = \lim_{n\to\infty} f_n$ in $L^2(\mathbb{R}_-, dq; \mathcal{H}^{-1})$ with $f_n \in L^2(\mathbb{R}^2_-, dqdp; \mathfrak{f})$, the sequence $u_n = (K_A - z)^{-1} f_n$ satisfies

$$\begin{aligned}&\gamma_{ev} u_n \in D(A) \quad , \quad \gamma_{odd} u_n = \mathrm{sign}\,(p) A \gamma_{ev} u_n\,, \\ &\lim_{n\to\infty} \|\gamma_{ev} u - \gamma_{ev} u_n\|_{L^2(\mathbb{R}, |p|dp; \mathfrak{f})} = 0\,, \\ &\lim_{n\to\infty} \|\,\mathrm{sign}\,(p)\gamma_{odd} u - A\gamma_{ev} u_n\|_{L^2(\mathbb{R}, |p|dp; \mathfrak{f})} = 0\,.\end{aligned}$$

Hence $\gamma_{ev} u \in D(A)$ and $\gamma_{odd} u = \mathrm{sign}\,(p) A \gamma_{ev} u$. $\square$

Proof of Proposition 2.3. By changing p into $-p$ and A into A^*, the operator K_{-,A^*} defined by

$$D(K_{-,A^*}) = \left\{ u \in L^2(\mathbb{R}_-, dq; \mathcal{H}^1)\,, \quad \begin{array}{l} P_- u \in L^2(\mathbb{R}^2_-, dqdp; \mathfrak{f})\,, \\ \gamma_{odd} u = -\,\mathrm{sign}\,(p) A^* \gamma_{ev} u \end{array} \right\}\,,$$

$$\forall u \in D(K_A^*)\,, \quad K_A^* u = P_- u = (-p\partial_q + \tfrac{-\partial_p^2 + p^2}{2})u\,,$$

is maximal accretive. Hence it suffices to prove $K_{-,A^*} \subset K_A^*$, which means

$$\forall u \in D(K_{-,A^*})\,,\ \forall v \in D(K_A)\,, \quad \langle K_{-,A^*}u\,,\, v\rangle = \langle u\,,\, K_A v\rangle\,. \qquad (32)$$

Both $u \in D(K_{-,A^*})$ and $v \in D(K_A)$ belong to $\mathcal{E}_{\mathbb{R}_-}(\mathfrak{f})$ and the polarized integration by parts formula of Proposition 2.10 is

$$\begin{aligned}
&\langle p\partial_q u\,,\, v\rangle + \langle u\,,\, p\partial_q v\rangle = \langle u\,,\, \mathrm{sign}\,(p)v\rangle_{L^2(\mathbb{R},|p|dp;\mathfrak{f})}\\
&\quad = \langle \mathrm{sign}\,(p)\gamma_{odd}u\,,\, \gamma_{ev}v\rangle_{L^2(\mathbb{R},|p|dp;\mathfrak{f})} + \langle \gamma_{ev}u\,,\, \mathrm{sign}\,(p)\gamma_{odd}v\rangle_{L^2(\mathbb{R},|p|dp;\mathfrak{f})}\\
&\qquad\qquad = \langle -A^*\gamma_{ev}u\,,\, \gamma_{ev}v\rangle_{L^2(\mathbb{R},|p|dp;\mathfrak{f})} + \langle \gamma_{ev}u\,,\, A\gamma_{ev}v\rangle_{L^2(\mathbb{R},|p|dp;\mathfrak{f})} = 0\,,
\end{aligned}$$

which yields (32). $\square$

3 Cuspidal semigroups

From the functional analysis point of view, the Kramers-Fokker-Planck operators have some specific properties, which were already used in [HerNi][EcHa] and [HelNi] in the case without boundary. Those properties behave well after tensorisation and are stable under relevant perturbations.

3.1 Definition and first properties

We work here in a complex Hilbert space $\mathcal{H}$ with the scalar product $\langle\,,\,\rangle$ and norm $\|\ \|$. A maximal accretive operator $(K, D(K))$ is a densely defined operator such

$$\forall u \in D(K)\,,\ \ \mathrm{Re}\,\langle u\,,\, Ku\rangle \geq 0\,,$$

and $1 + K : D(K) \to \mathcal{H}$ is a bijection. (see [Bre83][ReSi75][EnNa]). This is equivalent the fact that $(e^{-tK})_{t\geq 0}$ is a strongly continuous semigroup of contractions. This property passes to the adjoint $(K^*, D(K^*))$ (use $\|(\lambda + K^*)^{-1}\| = \|(\lambda+K)^{-1}\| \leq \frac{1}{\lambda}$ and Hille-Yosida theorem). The operator $(1+K)$ (resp. $1 + K^*$) defines isomorphisms in the diagram

$$\begin{aligned}
&D(K) \stackrel{1+K}{\to} \mathcal{H} \stackrel{1+K}{\to} D(K^*)'\,,\\
&D(K^*) \stackrel{1+K^*}{\to} \mathcal{H} \stackrel{1+K^*}{\to} D(K)'\,,
\end{aligned}$$

where $D(K^*)'$ (resp. $D(K)'$) is the dual of $D(K)$ (resp. $D(K^*)$) and the density of $D(K)$ (resp. $D(K^*)$) provides the embedding $\mathcal{H} \subset D(K)'$ (resp.

$\mathcal{H} \subset D(K^*)')$. The operator $(1+K^*)(1+K)$ is the self-adjoint positive operator defined as a Friedrichs extension with

$$D((1+K^*)(1+K)) = \{u \in D(K)\,,(1+K)^*(1+K)u \in \mathcal{H}\}\ .$$

The modulus $|1+K|$ is the square root, $|1+K| = \sqrt{(1+K^*)(1+K)}$. An accretive operator of which the closure is maximal accretive will be said essentially maximal accretive (see [HelNi]).

Definition 3.1. *Let $(\Lambda, D(\Lambda))$ be a positive self-adjoint operator such that $\Lambda \geq 1$ and let r belong to $(0,1]$. A maximal accretive operator $(K, D(K))$ is said (Λ, r)-cuspidal if $D(\Lambda)$ is dense in $D(K)$ endowed with the graph norm and if there exists a constant $C > 0$ such that*

$$\forall u \in D(\Lambda)\,, \quad \|Ku\| \leq C\|\Lambda u\|\,, \tag{33}$$

$$\forall u \in D(\Lambda)\,, \forall \lambda \in \mathbb{R}\,, \quad \|\Lambda^r u\| \leq C\left[\|(K-i\lambda)u\| + \|u\|\right]\,. \tag{34}$$

Definition 3.2. *We say that a maximal accretive operator $(K, D(K))$ is r-pseudospectral for some $r \in (0,1]$, if there exists $C_K > 0$ such that*

$$\forall \lambda \in \mathbb{R}, \quad \|(-1+i\lambda-K)^{-1}\| \leq C_K\langle\lambda\rangle^{-r}\,.$$

An easy reformulation of this definition is given by the

Proposition 3.3. *A maximal accretive operator $(K, D(K))$ is r-pseudospectral with $r \in (0,1]$, iff there exists $C_K' > 0$ such that*

- *the spectrum of K is contained in $S_K \cap \{\mathbb{R}\mathrm{e}\ z \geq 0\}$ with*

$$S_K = \left\{z \in \mathbb{C}\,, |z+1| \leq C_K' |\,\mathbb{R}\mathrm{e}\ z+1|^{1/r}\,, \mathbb{R}\mathrm{e}\ z \geq -1\right\}\ ;$$

- *for $z \notin S_K$ with $\mathbb{R}\mathrm{e}\ z \geq -1$, the resolvent norm is estimated by*

$$\|(z-K)^{-1}\| \leq C_K'\langle z\rangle^{-r}\,.$$

Proof. The "if" part is obvious. For the "only if" part it suffices to use the first resolvent identity. More precisely write

$$(K-\mu-i\lambda) = (K+1-i\lambda)\left[1-(1+\mu)(K+1-i\lambda)^{-1}\right]\,.$$

When $\langle\lambda\rangle^r > 2C_K|1+\mu|$ the right-hand side is invertible and $\|(K-\mu-i\lambda)^{-1}\| \leq 2\|(K+1-i\lambda)^{-1}\|$. ☐

Note that the case $r=1$ corresponds to the case of sectorial operators. Among other consequences of these properties, we shall prove that they are essentially equivalent (with some loss in the exponent).

Theorem 3.4. *For a maximal accretive operator $(K, D(K))$ the following statements satisfy*

$$(i) \Rightarrow (ii) \Rightarrow (iii)\,.$$

(i) *$(K, D(K))$ is (Λ, r)-cuspidal, $r \in (0,1]$.*

(ii) *$(K, D(K))$ is r-pseudospectral.*

(iii) *$(K, D(K))$ is $(|1+K|, r')$-cuspidal for $r' < \frac{r}{2-r}$.*

The first result taken from [HerNi] is about $(i) \Rightarrow (ii)$.

Proposition 3.5. *When $(K, D(K))$ is (Λ, r)-cuspidal with exponent $r \in (0,1]$, it is r-pseudospectral with the same exponent.*

Proof. For $u \in D(K)$, we start from

$$2\|(K-z)u\|^2 + 2|\,\mathbb{R}\mathrm{e}\ z|^2\|u\|^2 \geq \|(K - \mathrm{Im}\, z)u\|^2\,,$$

which gives

$$2\|(K-z)u\|^2 + (1+2|\,\mathbb{R}\mathrm{e}\ z|^2)\|u\|^2 \geq \|(K-\mathrm{Im}\, z)u\|^2 + \|u\|^2 \geq \frac{1}{2C}\|\Lambda^r u\|^2\,.$$

The condition (33) provides the operator inequality $1 \leq (1+K)^*(1+K) \leq \Lambda^2$. The operator monotonicity of $x \to x^r$ for $r \in [0,1]$ implies $1 \leq [(1+K^*)(1+K)]^r \leq \Lambda^{2r}$ and

$$\|(K-z)u\|^2 + (1+\mathbb{R}\mathrm{e}\ z)^2\|u\|^2 \geq \frac{1}{8C}\left[\|(K-z)u\|^2 + \langle u\,,\,[(1+K^*)(1+K)]^r u\rangle\right]\,.$$

Lemma B.1 of [HerNi] then says

$$4\left[\|(K-z)u\|^2 + \langle u\,,\,[(1+K^*)(1+K)]^r u\rangle\right] \geq |z+1|^{2r}\|u\|^2\,, \forall u \in D(K)\,,$$

as soon as $\mathbb{R}\mathrm{e}\ z \geq -1$. We have proved

$$\forall u \in D(K)\,, \|(K-z)u\|^2 \geq \left[\frac{1}{32C}|z+1|^{2r} - (1+\mathbb{R}\mathrm{e}\ z)^2\right]\|u\|^2\,,$$

for all $z \in \mathbb{C}$ such that $\mathbb{R}\mathrm{e}\; z \geq -1$. Any $z = (-1 + i\lambda)$ with $\lambda \in \mathbb{R}$ belongs to the resolvent set of K and we get the inequality

$$\|(K + 1 - i\lambda)^{-1}\| \leq 8\sqrt{C}\langle\lambda\rangle^{-r}\,.$$

$\square$

Remember the formula

$$e^{-tK} = \frac{1}{2i\pi}\int_{+i\infty}^{-i\infty} e^{-tz}(z-K)^{-1}\; dz\,,$$

which can be understood for general maximal accretive operators as the limit

$$\forall \psi \in D(K)\,,\; e^{-tK}\psi = \lim_{\varepsilon\to 0^+}\lim_{k\to\infty}\frac{1}{2i\pi}\int_{-\varepsilon+i\infty}^{-\varepsilon-i\infty}\frac{e^{-tz}}{1+\frac{z}{k}}(z-K)^{-1}\psi\; dz\,,$$

extended to any $\psi \in \mathcal{H}$ by density.
Proposition 3.3 allows a contour deformation which provides a norm convergent integral for $t > 0$.

Proposition 3.6. *Assume that $(K, D(K))$ is r-pseudospectral with exponent $r \in (0, 1]$. Let C_K' be the constant of Proposition 3.3 and let Γ_K be the contour $\left\{|Im\, z| = 2C_K|1 + \mathbb{R}\mathrm{e}\; z|^{\frac{1}{r}}\,, \mathbb{R}\mathrm{e}\; z \geq -1\right\}$ oriented from $+i\infty$ to $-i\infty$. Then for any $t > 0$*

$$e^{-tK} = \frac{1}{2i\pi}\int_{\Gamma_K} e^{-tz}(z-K)^{-1}\; dz \tag{35}$$

where the right-hand side is a norm convergent integral in $\mathcal{L}(\mathcal{H})$.

Proof. For $\psi \in D(K)$, $t > 0$ and $k \geq 2$ the function $\frac{e^{-tz}}{(1+\frac{z}{k})}(z-K)^{-1}\psi$ is a holomorphic function of z in $\{z \in \mathbb{C}\setminus S_K\,, \mathbb{R}\mathrm{e}\; z > -2\}$ with

$$\|\frac{e^{-tz}}{(1+\frac{z}{k})}(z-K)^{-1}\psi\| \leq C_k\frac{e^{-t\,\mathbb{R}\mathrm{e}\; z}}{\langle z\rangle^2}\|(1+K)\psi\|\,,$$

when $\mathbb{R}\mathrm{e}\; z \geq -1$, $z \in \mathbb{C}\setminus S_K$. The contour integral

$$\int_{-\varepsilon+i\infty}^{-\varepsilon-i\infty}\frac{e^{-tz}}{1+\frac{z}{k}}(z-K)^{-1}\psi\; dz\,,$$

for any $\varepsilon \in (0,1)$, can thus be deformed into

$$\int_{\Gamma_K} \frac{e^{-tz}}{1+\frac{z}{k}}(z-K)^{-1}\psi \; dz\,.$$

With the inequalities

$$\|(z-K)^{-1}\| \leq C'_K\langle z\rangle^{-r} \leq C'_K\langle \mathrm{Im}\, z\rangle^{-r}\,, \tag{36}$$

$$|e^{-tz}| = e^{-t\,\mathbb{R}\mathrm{e}\; z} \leq e^t e^{-t\frac{|\mathrm{Im}\, z|^r}{2C'_K}}\,,\; |dz| = \left[1+\mathcal{O}(|\mathrm{Im}\, z|^{2(r-1)})\right]|d\;\mathrm{Im}\, z| \tag{37}$$

valid for all $z \in \Gamma_K$, we can take the limit as $k \to \infty$ for any fixed $t > 0$, in the integral $\int_{\Gamma_K}$. The convergence holds for any $\psi \in \mathcal{H}$ and the integral (35) is a norm convergent integral. □

A classical results (see for instance [EnNa]) provides the equivalence between:

- $(e^{-zA})_{z\in C\cup\{0\}}$ is a bounded analytic semigroup for some open convex cone C.
- A is sectorial (see [EnNa] for a general definition).
- For all positive τ the estimate $\|(-\tau+is-A)^{-1}\| \leq \frac{C}{|s|}$ holds for all $s \neq 0$.
- The quantity $\sup_{t>0}\|tAe^{-tA}\|$ is finite.

Cuspidal semigroup, and this is a key idea of [HerNi], is a fractional version of the above notions. Actually Proposition 3.3 says that $(1+K)$ is sectorial when $r=1$. In this direction, the next result completes Proposition 3.6.

Proposition 3.7. *If $(K, D(K))$ is r-pseudospectral with exponent $r \in (0,1)$, then*

$$\sup_{t>0}\|t^{\frac{2}{r}-1}(1+K)e^{-t(1+K)}\| < +\infty\,. \tag{38}$$

If $(K, D(K))$ is (Λ, r)-cuspidal with $r \in (0,1)$, then

$$\sup_{t>0}\|t^{\frac{2}{r}-1}\Lambda^r e^{-t(1+K)}\| < +\infty\,. \tag{39}$$

Proof. For $u \in D(K)$ and $t > 0$, the formula (35) gives

$$e^{-tK}Ku = \frac{1}{2i\pi}\int_{\Gamma_K} e^{-tz}(z-K)^{-1}Ku\ dz = \frac{1}{2i\pi}\int_{\Gamma_K} e^{-tz}z(z-K)^{-1}u\ dz\,,$$

because $(z-K)^{-1}Ku = -u + z(z-K)^{-1}u$. From the inequalities (36) and (37) and $|z| \leq C''(1+|\mathrm{Im}\, z|)$ along Γ_K, we deduce

$$\|e^{-tK}Ku\| \leq C^{(3)}e^t \int_0^{+\infty}(1+\lambda)e^{-t\lambda^r}\lambda^{-r}\ d\lambda \leq C^{(4)}e^t\left[t^{1-\frac{1}{r}} + t^{1-\frac{2}{r}}\right]$$

The density of $D(K)$ yields

$$\forall t > 0\,, \quad e^{-t}\|Ke^{-t(1+K)}\| \leq C^{(4)}\left[t^{1-\frac{1}{r}} + t^{1-\frac{2}{r}}\right]e^{-t} \leq C^{(5)}t^{1-\frac{2}{r}}\,.$$

while we know $e^{-t}\|e^{-t(1+K)}\| \leq e^{-2t} \leq C_r t^{1-\frac{2}{r}}$. We have proved that $\sup_{t\geq 0}\|t^{\frac{2}{r}-1}(2+K)e^{-t(2+K)}\|$ is finite. Replacing K with $2K$, which is also r-pseudospectral owing to Proposition 3.3, and $2t$ by t, finishes the proof of (38).
When $(K, D(K))$ is (Λ, r)-cuspidal with $r < 1$, then it is r-pseudospectral according to Proposition 3.5. The second estimate then comes from

$$\|\Lambda^r u\| \leq C\left[\|Ku\| + \|u\|\right] \leq C'\|(1+K)u\|\,.$$

$\square$

Below is the converse implication of the second statement of Proposition 3.7.

Proposition 3.8. *Let $(K, D(K))$ be a maximal accretive operator. Assume that (39) is finite for $r \in (0,1]$ and assume the existence of an operator $\Lambda \geq 1$ such that $D(\Lambda)$ is dense in $D(K)$ and*

$$\forall u \in D(\Lambda)\,, \quad \|Ku\| \leq \|\Lambda u\|\,.$$

Then the operator $(K, D(K))$ is $(\Lambda, \theta r)$-cuspidal $\theta \in (0, \frac{r}{2-r}) \subset (0, r)$: there exists $C_\theta > 0$ such that

$$\forall u \in D(K)\,, \forall \lambda \in \mathbb{R}\,, \quad \|\Lambda^{\theta r}u\| \leq C_\theta\left[\|(K-i\lambda)u\| + \|u\|\right]\,.$$

Proof. For $t > 0$, $\lambda \in \mathbb{R}$ and $v \in \mathcal{H}$, the estimate

$$\|\Lambda^r e^{-t(1+K-i\lambda)}v\| \leq C_0 t^{1-\frac{2}{r}}\|v\|\,,$$

is interpolated into

$$\|\Lambda^{\theta r} e^{-t(1+K-i\lambda)}v\| \leq C_0^\theta t^{\theta\frac{r-2}{r}}\|v\|\,,$$

for any $\theta \in [0,1]$. Then the right-hand side of

$$\|\Lambda^{\theta r} e^{-t(K+2-i\lambda)}v\| \leq C_0^\theta e^{-t} t^{\theta\frac{r-2}{r}}\|v\|$$

is integrable on $(0,+\infty)$ as soon as $\theta\frac{r-2}{r} > -1$. From

$$(2+K-i\lambda)^{-1}v = \int_0^{+\infty} e^{-t(2+K-i\lambda)}v\ dt\,,$$

we deduce $(2+K-i\lambda)^{-1}v \in D(\Lambda^{r\theta})$ and

$$\|\Lambda^{r\theta}(2+K-i\lambda)^{-1}v\| \leq C_\theta\|v\|\,.$$

Setting $u = (2+K-i\lambda)^{-1}v$ gives

$$\|\Lambda^{r\theta}u\| \leq C_\theta\|2u + (K-i\lambda)u\| \leq C\left[\|(K-i\lambda)u\| + \|u\|\right]\,,$$

for all $u \in D(K)$, as soon as $\theta \in (0, \frac{r}{2-r})$. $\square$

Proof of (*ii*) $\Rightarrow$ (*iii*) *in Theorem 3.4.* Firstly, $D(|1+K|)$ is the form domain of $(1+K^*)(1+K)$ and equals $D(K)$, with

$$\forall u \in D(K) \quad \|Ku\|^2 \leq \|(1+K)u\|^2 = \||1+K|u\|^2\,.$$

The end is a variant of the previous argument. The relation (38) can be written

$$\||1+K|e^{-t(1+K)}\| \leq Ct^{1-\frac{2}{r}}\,,$$

and leads to

$$\forall \lambda \in \mathbb{R}\,, \forall v \in \mathcal{H}\,, \||1+K|^\theta e^{-t(2+K-i\lambda)}\| \leq C_\theta e^{-t} t^{\theta\frac{r-2}{r}}\|v\|\,.$$

By taking $\theta < \frac{r}{2-r}$, we deduce $(2+K-i\lambda)^{-1}v \in D(|1+K|^\theta)$ and

$$\||1+K|^\theta(2+K-i\lambda)^{-1}v\| \leq C'_\theta\|v\|\,.$$

With $u = (2+K-i\lambda)^{-1}v$ this means

$$\forall \lambda \in \mathbb{R}\,, \forall u \in D(K)\,, \quad \||1+K|^\theta u\| \leq C''_\theta\left[\|(K-i\lambda)u\| + \|u\|\right]\,.$$

$\square$

3.2 Perturbation

The Theorem X.50 of [ReSi75] says that a pair of accretive operators A and C defined on a same dense domain $D \subset \mathcal{H}$, such that

$$\forall u \in D\,, \quad \|(A-C)u\| \leq a\left[\|Au\| + \|Cu\|\right] + b\|u\|\,,$$

for some fixed $a < 1$ and $b > 0$, satisfy :

- the closures $\overline{A}$ and $\overline{C}$ have the same domain;
- $\overline{A}$ is maximal accretive if and only if $\overline{C}$ is.

Of course this applies to $C = A + B$ when B is a relatively bounded perturbation

$$\forall u \in D(A)\,, \quad \|Bu\| \leq a\|Au\| + b\|u\|\,.$$

Below is the cuspidal version, which requires a uniform control with respect to $i\lambda \in i\mathbb{R}$.

Proposition 3.9. *Let $(K, D(K))$ be a (Λ, r)-cuspidal operator with exponent $r \in (0,1]$. Assume that B is a relatively bounded perturbation of K such that*

$$\forall u \in D(K)\,, \forall \lambda \in \mathbb{R}\,, \quad \|Bu\| \leq a\|(K-i\lambda)u\| + b\|u\|$$

with some fixed $a < 1$ and $b > 0$. If $(K+B)$ is accretive on $D(K)$, then $(K+B, D(K))$ is (Λ, r)-cuspidal.
The same statement holds for r-pseudospectral operators with exponent $r \in (0,1]$.

Proof. The Theorem X.50 of [ReSi75] says that $(K+B, D(K))$ is maximal accretive.
For the inequalities, write simply for $u \in D(\Lambda) \subset D(K)$ and $\lambda \in \mathbb{R}$,

$$\begin{aligned}
\|(K+B)u\| &\leq \|Ku\| + \|Bu\| \leq (1+a)\|Ku\| + b\|u\| \leq [C(1+a)+b]\|\Lambda u\|\,,\\
\|(K+B-i\lambda)u\| &\geq \|(K-i\lambda)u\| - \|Bu\| \geq (1-a)\|(K-i\lambda)u\| - b\|u\|\\
&\geq (1-a)C^{-1}\|\Lambda^r u\| - (b+1)\|u\|\,.
\end{aligned}$$

This yields

$$\forall \lambda \in \mathbb{R}\,, \forall u \in D(\Lambda)\,, \|Ku\| \leq C'\|\Lambda u\| \quad , \quad \|\Lambda^r u\| \leq C'\left[\|(K-i\lambda)u\| + \|u\|\right]\,,$$

with $C' = \max\left\{C(1+a)+b, \frac{C(b+1)}{1-a}\right\}$.
When $(K, D(K))$ is r-pseudospectral, the same lower bound $\|(K+B-i\lambda)u\| \geq (1-a)\|(K-i\lambda)u\| - b\|u\|$ allows to conclude that $(K+B, D(K))$ is r-pseudospectral. $\square$

Corollary 3.10. *A maximal accretive operator $(K, D(K))$ is (Λ, r)-cuspidal if and only if there exists a constant $C \in \mathbb{R}$ such that $(C+K, D(K))$ is (Λ, r)-cuspidal.*
If $(K, D(K))$ is (Λ, r)-cuspidal with exponent $r \in (0,1]$ and B is defined on $D(K)$ with

$$\forall u \in D(K)\,, \|Bu\|^2 \leq C_B \,\mathbb{R}\mathrm{e}\,\langle u\,,\,(1+K)u\rangle$$

and $K+B$ accretive. Then $(K+B, D(K))$ is (Λ, r)-cuspidal.
The same statements hold for r-pseudospectral operators.

Proof. The first result comes readily from

$$\|-Cu\| \leq 0 \times \|(K-i\lambda)u\| + |C| \times \|u\|\,.$$

The second one comes from

$$\begin{aligned}\|Bu\|^2 &\leq C_B \,\mathbb{R}\mathrm{e}\,\langle u\,,\,(K-i\lambda)u\rangle \leq C_B\|u\|\|(K-i\lambda)u\| \\ &\leq \left[\frac{1}{2}\|(K-i\lambda)u\| + C_B\|u\|\right]^2.\end{aligned}$$

$\square$

3.3 Tensorisation

Take two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ and consider the Hilbert tensor product $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. We first recall a result about maximal accretivity.

Proposition 3.11. *Assume that $(K_1, D(K_1))$ and $(K_2, D(K_2))$ are maximal accretive in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Then the closure of $(K_1 \otimes Id_{\mathcal{H}_2} + Id_{\mathcal{H}_1} \otimes K_2)$ initially defined on the algebraic tensor product $D(K_1) \overset{alg}{\otimes} D(K_2)$, is maximal accretive.*

Then we will prove the cuspidal version.

Proposition 3.12. *Assume that $(K_1, D(K_1))$ and $(K_2, D(K_2))$ are respectively (Λ_1, r_1) and (Λ_2, r_2)-cuspidal operators in $\mathcal{H}_1$ and $\mathcal{H}_2$. Then the closure of $(K_1 \otimes Id_{\mathcal{H}_2} + Id_{\mathcal{H}_1} \otimes K_2)$ initially defined on the algebraic tensor product $D(K_1) \overset{alg}{\otimes} D(K_2)$, is (Λ, r)-cuspidal with*

$$\Lambda = \Lambda_1 \otimes Id + Id \otimes \Lambda_2\,,$$
$$\text{and} \qquad r < \min\{\frac{r_1^2}{8-2r_1}, \frac{r_2^2}{8-2r_2}\}\,.$$

Proof of Proposition 3.11 (for the sake of completeness). Consider the strongly semigroup of contractions $(e^{-tK_1} \otimes e^{-tK_2})_{t\geq 0}$ on $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ and call $(K, D(K))$ its maximal accretive generator. The operator $(K, D(K))$ is the closure of $(K_1 \otimes \mathrm{Id}_{\mathcal{H}_2} + \mathrm{Id}_{\mathcal{H}_1} \otimes K_2)$ defined on $D = D(K_1) \overset{alg}{\otimes} D(K_2)$: Indeed, for every pair $(\varphi_1, \varphi_2) \in D(K_1) \times D(K_2)$, $\mathbb{R}_+ \ni t \mapsto e^{-tK}(\varphi_1 \otimes \varphi_2) = (e^{-tK_1}\varphi_1) \otimes (e^{-tK_2}\varphi_2)$ is a $\mathcal{C}^1$-function. This implies $D \subset D(K)$. For $s > 0$, the expression

$$A_s\varphi = \frac{1}{s}\int_0^s e^{-t(1+K)}\varphi\, dt\,,$$

defines a continuous operator from $\mathcal{H}$ to $D(K)$. For $\varphi \in D(K)$, the relation

$$(1+K)\varphi - (1+K)A_s\varphi = (1+K)\varphi + \frac{1}{s}\left(e^{-s(K+1)}\varphi - \varphi\right)$$

implies

$$\|\varphi - A_s\varphi\|_{D(K)} = \|(1+K)(\varphi - A_s\varphi)\|_{\mathcal{H}} \overset{s\to 0^+}{\to} 0\,.$$

Since $D(K_j)$ is dense in $\mathcal{H}_j$ for $j = 1, 2$, φ can be approximated in $\mathcal{H}$ by an element $\sum_{j=1}^J \varphi_{1,j} \otimes \varphi_{2,h}$ of D with

$$\|A_s\left(\varphi - \sum_{j=1}^J \varphi_{1,j} \otimes \varphi_{2,h}\right)\|_{D(K)} \leq C_s\|\varphi - \sum_{j=1}^J \varphi_{1,j} \otimes \varphi_{2,h}\|_{\mathcal{H}}\,.$$

Now the integral

$$A_s\left(\sum_{j=1}^J \varphi_{1,j} \otimes \varphi_{2,h}\right) = \sum_{j=1}^J \frac{1}{s}\int_0^s (e^{-tK_1}\varphi_{1,j}) \otimes (e^{-tK_2}\varphi_{2,j})\, dt$$

can be approximated in $D(K)$ by a Riemann sum which belongs to D. Hence D is dense in $D(K)$ endowed with the graph norm $\|u\|_{D(K)} = \|(1+K)u\|_{\mathcal{H}}$. It is a core for $(K, D(K))$. $\square$

Proof of Proposition 3.12. We know from Proposition 3.11 that $K = K_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes K_2$ is essentially maximal accretive on $D(K_1) \otimes^{alg} D(K_2)$.
Let us prove the cuspidal property. The operator $\Lambda_j \geq 1$, $j = 1, 2$, is a self-adjoint operator in $\mathcal{H}_j$ with $D(\Lambda_j)$ dense in $D(K_j)$ and such that

$$\|K_j u\|_{\mathcal{H}_j} \leq C_j \|\Lambda_j u\|_{\mathcal{H}_j} \quad , \quad \|\Lambda_j^{r_j} u\|_{\mathcal{H}_j} \leq C_j \left[\|(K_j - i\lambda)u\|_{\mathcal{H}_j} + \|u\|_{\mathcal{H}_j}\right] ,$$

hold for any $u \in D(\Lambda_j)$ and $\lambda \in \mathbb{R}$. Call $\Lambda = \Lambda_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes \Lambda_2$ the essentially self-adjoint operator on $D(\Lambda_1) \otimes^{alg} D(\Lambda_2)$. Its domain $D(\Lambda)$ is dense in $D(K)$, and

$$\forall u \in D(\Lambda) , \quad \|Ku\| \leq C\|\Lambda u\| ,$$

is obtained after taking first $u \in D(\Lambda_1) \otimes^{alg} D(\Lambda_2)$.
The inequality (39) applied for $j = 1, 2$ gives the uniform bound

$$\|t^{\frac{2}{r_j}-1} \Lambda_j^{r_j} e^{-t(1+K_j)}\|_{\mathcal{L}(\mathcal{H}_j)} \leq C'_j .$$

This yields the uniform bound

$$\|(\Lambda_1^{r_1} + \Lambda_2^{r_2}) e^{-t(2+K)}\|_{\mathcal{L}(\mathcal{H})} \leq \max\{C'_1 t^{1-\frac{2}{r_1}}, C'_2 t^{1-\frac{2}{r_2}}\} e^{-t}$$

We take $\varrho = \min\{r_1, r_2\}$ and the functional calculus of commuting self-adjoint operators gives

$$\begin{aligned} &\Lambda^{2\varrho} \leq C_{r_1,r_2} (\Lambda_1^{r_1} + \Lambda_2^{r_2})^2 , \\ \text{and} \quad &\|\Lambda^{\varrho} e^{-t(2+K)}\| \leq C'_{r_1,r_2} t^{2-\frac{4}{\varrho}} e^{-t} , \end{aligned}$$

By interpolation we deduce

$$\sup_{t>0} \|t^{\frac{2}{\varrho}-1} \Lambda^{\frac{\varrho}{2}} e^{-t(2+K)}\| < +\infty .$$

Proposition 3.8 implies that $(K, D(K))$ is (Λ, r)-cuspidal for any r in $(0, \frac{\varrho^2}{8-2\varrho})$. $\square$

4 Separation of variables

The results of Subsection 3.3 provide a strategy for studying the maximal accretivity and cuspidal property for operators which make possible a complete separation of variables. Half-space problems associated with differential operators with separated variables can be reduced to the one dimensional half-line problem when the boundary conditions agree with this separation of variables. We focus on the cases when the boundary condition is given by $\gamma_{odd} u = \nu \,\mathrm{sign}\,(p)\gamma_{ev} u$ with $\nu \in \{0, 1\}$.

Although Proposition 3.12 provides the guideline, we do not apply it naively. Instead, we develop an analysis of abstract one-dimensional problems which will lead in the next section to more accurate results. In particular, we specify the domains of the operators which are implicit Proposition 3.11 and Proposition 3.12. Then we solve inhomogeneous boundary value problems by using variational arguments inspired by [Lio][Bar][Luc][Car] in the case $\nu = 1$.

4.1 Some notations

In this section, we assume the operators $(L_\pm, D(L_\pm))$ to be maximal accretive in the separable Hilbert space $\mathfrak{L}$ with $L_\pm^* = L_\mp$. For $I = \mathbb{R}_-$ or $I = \mathbb{R}$, we shall consider the operator

$$P_\pm^L = \pm p.\partial_q + \frac{-\partial_p^2 + p^2 + 1}{2} + L_\pm = \pm p\partial_q + \frac{1}{2} + \mathcal{O} + L_\pm$$

defined with the proper domain, containing $\mathcal{C}_0^\infty(I \times \mathbb{R}; D(L_\pm))$ and specified below, in $L^2(I \times \mathbb{R}, dqdp; \mathfrak{L}) = L^2(I \times \mathbb{R}, dqdp) \otimes \mathfrak{L}$. The formal adjoint of $P_\pm^L$ is $P_\mp^L$. Both operators $1 + L_\pm$ are isomorphisms in the diagram

$$D(L_\pm) \overset{1+L_\pm}{\to} \mathfrak{L} \overset{1+L_\pm}{\to} D(L_\pm^*)' = D(L_\mp)'.$$

where $D(L_\pm^*)'$ is the dual of $D(L_\pm^*)$ and the density of $D(L_\pm)$ provides the embedding $\mathfrak{L} \subset D(L_\pm^*)'$. In particular $P_\pm^L$ defines a continuous operator from $L^2(I \times \mathbb{R}, dqdp; \mathfrak{L})$ to $\mathcal{D}'(I \times \mathbb{R}; D(L_\mp)')$ for any open interval $I \subset \mathbb{R}$.
The densely defined quadratic form $\mathrm{Re}\,\langle u\,,\, L_\pm u\rangle$ defines a self-adjoint operator (take the Friedrichs extension see [ReSi75]-Theorem X.23). This operator is (abusively) denoted by $\mathrm{Re}\, L$. For any $u \in D(L_+) \cap D(L_-)$, $\mathrm{Re}\, Lu$ equals $\frac{L_+ + L_-}{2} u$. The notation $\mathrm{Re}\, L$ is especially justified under the following assumption.

Hypothesis 1. *The intersection $D(L_\pm) \cap D(L_-)$ is dense in $D(L_\pm)$ endowed with its graph norm and $\mathrm{Re}\, L$ is essentially self-adjoint on $D(L_+) \cap D(L_-)$.*

Under this assumption, $D(L_+)$, $D(L_-)$ and $D(L_+)\cap D(L_-)$ are densely included in $D((\mathbb{R}\mathrm{e}\ L)^{1/2})$, which is the quadratic form domain of $\mathbb{R}\mathrm{e}\ L$, and we have the embeddings

$$D(L_\pm)\subset D((\mathbb{R}\mathrm{e}\ L)^{1/2})\subset \mathfrak{L}\subset D((\mathbb{R}\mathrm{e}\ L)^{1/2})'\subset D(L_\mp)'\,.$$

Contrary to the case when $(L_+, D(L_+))$ is a non negative self-adjoint operator, there is no reason to assume in general that L_+ is continuous from $D((\mathbb{R}\mathrm{e}\ L)^{1/2})$ to $D((\mathbb{R}\mathrm{e}\ L)^{1/2})'$. This would mean that $\mathrm{Im}\, L=\frac{L_+-L_-}{2i}$ is estimated in terms of $\mathbb{R}\mathrm{e}\ L$.
According to this multiplicity of spaces, the notations of Section 2 will be adapted with various choices for the Hilbert space $\mathfrak{f}$. This will be specified in every case. For example $\mathcal{H}^s$, $s\in\mathbb{R}$, will follow the definition $\mathcal{H}^s=(\frac{1}{2}+\mathcal{O})^{-s/2}L^2(\mathbb{R},dp;\mathbb{C})$ of Section 2.1 with $\mathfrak{f}=\mathbb{C}$.
We shall use the following space

$$\mathcal{H}^{1,L}=\left\{u\in L^2(\mathbb{R},dp;\mathfrak{L})\,,\|u\|^2_{\mathcal{H}^1}+\langle u\,,\,(\mathbb{R}\mathrm{e}\ L)u\rangle_{\mathfrak{L}}<+\infty\right\}\,, \tag{40}$$

endowed with its natural norm and scalar product. Its dual is denoted by $\mathcal{H}^{-1,L}$. When L is bounded in $\mathfrak{L}$, it is nothing but $\mathcal{H}^{1,0}=\mathcal{H}^1\otimes\mathfrak{L}$ and $\mathcal{H}^{-1,0}=\mathcal{H}^{-1}\otimes\mathfrak{L}$. For an interval I of $\mathbb{R}$

$$\begin{aligned}L^2(I,dq;\mathcal{H}^1\otimes D(L_\pm))\subset L^2(I,dq;\mathcal{H}^{1,L})\subset L^2(I\times\mathbb{R},dqdp;\mathfrak{L})\\ \subset L^2(I,dq;\mathcal{H}^{-1,L})\subset L^2(I,dq;\mathcal{H}^{-1}\otimes D(L_\mp)')\,.\end{aligned}$$

The odd and even part of elements of $L^2(\mathbb{R},|p|dp;\mathfrak{L})$ involves an involution j acting on $\mathfrak{L}$. In the end, this construction will be applied with $\mathfrak{L}=\mathfrak{L}_1\otimes\mathfrak{f}_1$ with $L=L_1\otimes\mathrm{Id}$ and $j=\mathrm{Id}\otimes j_1$.

Definition 4.1. *The Hilbert space $\mathfrak{L}$ is endowed with a unitary involution j which commutes with $e^{-tL_\pm}$ for all $t\geq 0$. In $L^2(\mathbb{R},|p|dp;\mathfrak{L})$ or $\mathcal{D}'(\mathbb{R}^*;\mathfrak{f})$ with $\mathfrak{f}\in\{D(L_\pm),\mathfrak{L},D(L_\pm)'\}$, the even and odd part are given by*

$$\gamma_{ev}(p)=[\Pi_{ev}\gamma](p)=\frac{\gamma(p)+j\gamma(-p)}{2}\,, \tag{41}$$

$$\gamma_{odd}(p)=[\Pi_{odd}\gamma](p)=\frac{\gamma(p)-j\gamma(-p)}{2}\,. \tag{42}$$

The operators Π_+ and Π_- are given by

$$\Pi_+=\Pi_{ev}+sign\,(p)\Pi_{odd}\quad,\quad \Pi_-=\Pi_{ev}-sign\,(p)\Pi_{odd}\,, \tag{43}$$

The projections Π_{ev} and Π_{odd} are orthogonal in $L^2(\mathbb{R},|p|dp;\mathfrak{L})$ with

$$\begin{aligned}
&\Pi_{ev}^* = \Pi_{ev} = (1-\Pi_{odd})\,,\\
&\mathrm{sign}\,(p)\circ\Pi_{ev} = \Pi_{odd}\circ\mathrm{sign}\,(v)\,, \qquad (44)\\
&\int_{\mathbb{R}}\langle\gamma(p)\,,\,\gamma'(p)\rangle_{\mathfrak{L}}\;pdp = \langle\gamma\,,\,\mathrm{sign}\,(p)\gamma'\rangle_{L^2(\mathbb{R},|p|dp;\mathfrak{L})}\\
&\quad = \langle\gamma_{ev}\,,\,\mathrm{sign}\,(p)\gamma'_{odd}\rangle_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} + \langle\gamma_{odd}\,,\,\mathrm{sign}\,(p)\gamma'_{ev}\rangle_{L^2(\mathbb{R},|p|dp;\mathfrak{L})}\,. \qquad (45)
\end{aligned}$$

Moreover the two operators Π_+ and Π_- are projections owing to $\mathrm{sign}\,(p)\Pi_{ev} = \Pi_{odd}\,\mathrm{sign}\,(p)$, with the same range $\mathrm{Ran}\,\Pi_+ = \mathrm{Ran}\,\Pi_- = \mathrm{Ran}\,\Pi_{ev}$. With $\Pi_{ev} = \frac{\Pi_+ + \Pi_-}{2}$ and $\mathrm{sign}\,(p)\Pi_{odd} = \frac{\Pi_+ - \Pi_-}{2}$, the quantity (45) also equals

$$\frac{1}{2}\left[\langle\Pi_+\gamma\,,\,\Pi_+\gamma'\rangle_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} - \langle\Pi_-\gamma\,,\,\Pi_-\gamma'\rangle_{L^2(\mathbb{R},|p|dp;\mathfrak{L})}\right]\,. \qquad (46)$$

The projection Π_+ (resp. Π_-) has the same kernel as $1_{\mathbb{R}_+}(p)$ (resp. $1_{\mathbb{R}_-}(p)$) according to the explicit formulas:

$$\Pi_+\gamma = (1_{\mathbb{R}_+}(p) + j1_{\mathbb{R}_-}(p))\gamma(|p|) \quad , \quad \Pi_-\gamma = (j1_{\mathbb{R}_+}(p) + 1_{\mathbb{R}_-}(p))\gamma(-|p|)\,.$$

The mapping $\Pi_+\times\Pi_- : L^2(\mathbb{R},|p|dp;\mathfrak{L}) \to (\mathrm{Ran}\,\Pi_{ev})^2$ is an isomorphism

$$\begin{aligned}
&(\Pi_+,\Pi_-) : L^2(\mathbb{R},|p|dp;\mathfrak{L}) \leftrightarrows (\mathrm{Ran}\,\Pi_{ev})^2 \qquad (47)\\
\text{with}\quad & L^2(\mathbb{R},|p|dp;\mathfrak{L}) = \ker(\Pi_+)\overset{\perp}{\oplus}\ker(\Pi_-)\,,\\
\text{and}\quad & (\Pi_+,\Pi_-)\gamma_{ev} = (\gamma_{ev},\gamma_{ev})\,,\\
& (\Pi_+,\Pi_-)\gamma_{odd} = (\mathrm{sign}\,(p)\gamma_{odd}, -\,\mathrm{sign}\,(p)\gamma_{odd})\,.
\end{aligned}$$

4.2 Traces and integration by parts

For a given interval, we shall consider the space

$$\mathcal{E}_I^{L,\pm} = \left\{u\in L^2(I,dq;\mathcal{H}^{1,L})\,,\, P_\pm^L u \in L^2(I,dq;\mathcal{H}^{-1,L})\right\}\,,$$

endowed with the norm

$$\|u\|_{\mathcal{E}_I^{L,\pm}} = \|u\|_{L^2(I,dq;\mathcal{H}^{1,L})} + \|P_\pm^L u\|_{L^2(I,dq;\mathcal{H}^{-1,L})}\,.$$

Proposition 4.2. *Under Hypothesis 1 the space $\mathcal{E}_I^{L,\pm}$ is continuously embedded in the space $\mathcal{E}_I(D(L_\mp)')$ of Definition 2.9 with $\mathfrak{f} = D(L_\mp)'$, and therefore in $\mathcal{C}_b^0(\overline{I};L^2(\mathbb{R},|p|dp;D(L_\mp)')$.*

Proof. From the embeddings $\mathcal{H}^{1,L} \subset \mathcal{H}^{1,0} = \mathcal{H}^1 \otimes \mathfrak{L} \subset \mathcal{H}^1 \otimes D(L_\mp)'$ and $\mathcal{H}^{-1,L} \subset \mathcal{H}^{-1} \otimes D(L_\mp)'$, we deduce

$$\begin{aligned} & u \in L^2(I, dq; \mathcal{H}^1 \otimes \mathfrak{L}) \subset L^2(I, dq; \mathcal{H}^1 \otimes D(L_\mp)')\,, \\ \text{and} \qquad & p\partial_q u = \pm \left[P_\pm^L u - (\frac{1}{2} + \mathcal{O})u - L_\pm u \right] \in L^2(I, dq; \mathcal{H}^{-1} \otimes D(L_\mp)')\,, \end{aligned}$$

because $u \in L^2(I, dq; \mathcal{H}^1 \otimes \mathfrak{L})$ implies $L_\pm u \in L^2(I, dq; \mathcal{H}^1 \otimes D(L_\mp)')$ and $(\frac{1}{2} + \mathcal{O})u \in L^2(I, dq; \mathcal{H}^{-1} \otimes \mathfrak{L})$. We conclude by referring to Proposition 2.10 with $\mathfrak{f}$ replaced by $D(L_\mp)'$. $\square$

Proposition 4.3. *Assume Hypothesis 1 and for $I = \mathbb{R}_-$ set $\gamma u = u(q = 0)$ for $u \in \mathcal{E}_{\mathbb{R}_-}^{L,\pm}$. The integration by part formula*

$$\begin{aligned} & \langle v\,,\, P_+^L u\rangle = \lim_{\varepsilon \to 0^+} \langle \gamma v_\varepsilon\,,\, sign\,(p) \gamma u_\varepsilon \rangle_{L^2(\mathbb{R}, |p|dp; \mathfrak{L})} + \langle P_-^L v\,,\, u\rangle\,, \\ & v_\varepsilon = (1 + \varepsilon L_-)^{-1} v \quad , \quad u_\varepsilon = (1 + \varepsilon L_+)^{-1} u\,, \end{aligned}$$

holds for pairs $(u, v) \in \mathcal{E}_{\mathbb{R}_-}^{L,+} \times \mathcal{C}_0^\infty((-\infty, 0] \times \mathbb{R}; D(L_-))$ and for pairs (u, v) which satisfy

- *u and v belong to $L^2(\mathbb{R}_-, dq, \mathcal{H}^{1,L})$;*
- *$P_+^L u$ and $P_-^L v$ belong to $L^2(\mathbb{R}_-, dq; \mathcal{H}^{-1} \otimes \mathfrak{L})$.*

Proof. In both cases, Proposition 4.2 ensures $u \in \mathcal{E}_{\mathbb{R}_-}(D(L_-)')$ and $v \in \mathcal{E}_{\mathbb{R}_-}(D(L_+)')$.
When $(u, v) \in \mathcal{E}_{\mathbb{R}_-}^{L,+} \times \mathcal{C}_0^\infty((-\infty, 0] \times \mathbb{R}; D(L_-)))$, the integration by part

$$\begin{aligned} \langle v\,,\, p\partial_q u + (\frac{1}{2} + \mathcal{O})u\rangle = & \int_{\mathbb{R}} \langle v(0, p)\,,\ u(0, p)\rangle_{D(L_-)\,,\,D(L_-)'}\, pdp \\ & + \langle (-p\partial_q v + \frac{1}{2} + \mathcal{O})v\,,\, u\rangle \end{aligned}$$

makes sense, while the other term of $P_+^L u$ gives the integrable quantities

$$\langle v(q, p)\,,\, L_+ u(q, p)\rangle_{D(L_-)\,,\,D(L_-)'} = \langle L_- v(q, p)\,,\, u(q, p)\rangle\,.$$

In the second case $u_\varepsilon = (1 + \varepsilon L_+)^{-1} u$ and $v_\varepsilon = (1 + \varepsilon L_-)^{-1} v$ both belong to $\mathcal{E}_{\mathbb{R}_-}(\mathfrak{L})$ and we can apply the polarized integration by part of Proposition 2.10

$$\langle p\partial_q u_\varepsilon\,,\, v_\varepsilon\rangle + \langle u_\varepsilon\,,\, p\partial_q v_\varepsilon\rangle = \int_{\mathbb{R}} \langle u_\varepsilon(0, p)\,,\, v_\varepsilon(0, p)\rangle_{\mathfrak{L}}\, pdp\,.$$

The term with $(\frac{1}{2}+\mathcal{O})$ involves only the duality between $\mathcal{H}^1$ and $\mathcal{H}^{-1}$ while the terms with L_+ and L_- is concerned with $L^2(\mathbb{R}^2_-, dqdp;\mathfrak{L})$ functions. We obtain

$$\begin{aligned}\langle (1+\varepsilon L_-)^{-1}v\,,\,(1+\varepsilon L_+)^{-1}P_+^L u\rangle - \langle (1+\varepsilon L_-)^{-1}P_-^L v\,,\,(1+\varepsilon L_+)^{-1}u\rangle \\ = \langle (1+\varepsilon L_-)^{-1}\gamma v\,,\,\mathrm{sign}\,(p)(1+\varepsilon L_+)^{-1}\gamma u\rangle_{L^2(\mathbb{R},|p|dp;\mathfrak{L})}\end{aligned}$$

Our assumptions were made so that the left-hand side, and therefore the right-hand side, converge. In particular $P_+^L u\,,\,P_-^L v\in L^2(\mathbb{R}_-, dq;\mathcal{H}^{-1}\otimes\mathfrak{L})$ is used with $\mathrm{s}-\lim_{\varepsilon\to 0}(1+\varepsilon L_\pm)^{-1}=1$ in $\mathfrak{L}$ and $\|(1+\varepsilon L_\pm)^{-1}\|_{\mathcal{L}(\mathfrak{L})}\leq 1$. ☐

Remark 4.4. *The last argument cannot be used for general* $(u,v)\in\mathcal{E}^{L,+}_{\mathbb{R}_-}\times\mathcal{E}^{L,-}_{\mathbb{R}_-}$ *because we do not control the effect of the regularization* $(1+\varepsilon L_\pm)^{-1}$ *on* $D((\mathrm{Re}\ L)^{1/2})$ *and* $\mathcal{H}^{\pm 1,L}$.

Proposition 4.5. *Assume Hypothesis 1. Take* $I=(a,b)$ *with* $-\infty\leq a<b\leq+\infty$. *Assume* $u\in\mathcal{E}_I^{L,\pm}$ *with* $P_\pm^L u\in L^2(I,dq;\mathcal{H}^{-1}\otimes\mathfrak{L})$ *and* $\gamma_a u=u(a,.)\in L^2(\mathbb{R},|p|dp;\mathfrak{L})$ *if* $-\infty<a$ *and* $\gamma_b u=u(b,.)\in L^2(\mathbb{R},|p|dp;\mathfrak{L})$ *if* $b<+\infty$. *Then the following inequality holds*

$$\begin{aligned}2\|u\|^2_{L^2(I,dq;\mathcal{H}^{1,L})}\leq 2\,\mathrm{Re}\,\langle u\,,\,P_\pm^L u\rangle \pm 1_{\mathbb{R}}(a)\int_{\mathbb{R}}|u(a,p)|^2_{\mathfrak{L}}\ pdp \\ \mp 1_{\mathbb{R}}(b)\int_{\mathbb{R}}|u(b,p)|^2_{\mathfrak{L}}\ pdp\,.\end{aligned}$$

Proof. With an obvious change of signs, it suffices to consider the $+$ case. From Proposition 4.2 we know $u\in\mathcal{E}_I(D(L_-)')$ and the traces are well defined. Set $f=P_+^L u$, $f_\partial^a=\gamma_a u$ and $f_\partial^b=\gamma_b u$. For $\varepsilon>0$ take $u_\varepsilon=(1+\varepsilon L_+)^{-1}u$ and set $f_\varepsilon=(1+\varepsilon L_+)^{-1}f$, $f_{\partial,\varepsilon}^{a,b}=(1+\varepsilon L_+)^{-1}f_\partial^{a,b}$. Then u_ε belongs to $L^2(I,dq;\mathcal{H}^1\otimes D(L_+))\subset L^2(I,dq;\mathcal{H}^1\otimes\mathfrak{L})$ and satisfies

$$\begin{aligned}p\partial_q u_\varepsilon &= f_\varepsilon-(\frac{1}{2}+\mathcal{O})u_\varepsilon - L_+u_\varepsilon \quad \in L^2(I,dq;\mathcal{H}^{-1}\otimes\mathfrak{L})\,,\\ \gamma_{a,b}u_\varepsilon &= f_{\partial,\varepsilon}^{a,b}\in L^2(\mathbb{R},|p|dp;\mathfrak{L})\,.\end{aligned}$$

Hence the integration by parts formula of Proposition 2.10 can be used with $\mathfrak{f}=\mathfrak{L}$:

$$\begin{aligned}1_{\mathbb{R}}(b)\int_{\mathbb{R}}|f^b_{\partial,\varepsilon}(p)|^2_{\mathfrak{L}}\ pdp - 1_{\mathbb{R}}(a)\int_{\mathbb{R}}|f^a_{\partial,\varepsilon}(p)|^2_{\mathfrak{L}}\ pdp \quad &= 2\,\mathrm{Re}\,\langle u_\varepsilon\,,\,p\partial_q u_\varepsilon\rangle \\ &= 2\,\mathrm{Re}\,\langle u_\varepsilon\,,\,f_\varepsilon\rangle - 2\|u_\varepsilon\|^2_{L^2(I,dq;\mathcal{H}^{1,L})}\,.\end{aligned}$$

With $\|(1+\varepsilon L_+)^{-1}\|_{\mathcal{L}(\mathfrak{L})} \leq 1$ and $\mathcal{H}^{1,0} = \mathcal{H}^1 \otimes \mathfrak{L}$, we deduce from this the inequalities

$$\begin{aligned}
\|u_\varepsilon\|^2_{L^2(I,dq;\mathcal{H}^{1,0})} + \|u_\varepsilon\|^2_{L^2(I,dq;\mathcal{H}^{1,L})} &\leq 2\|u_\varepsilon\|^2_{L^2(I,dq;\mathcal{H}^{1,L})} \\
&\leq \|u_\varepsilon\|_{L^2(I,dq;\mathcal{H}^{1,0})}\|f\|_{L^2(I,dq;\mathcal{H}^{-1,0})} + 1_{\mathbb{R}}(b)\|f^b_\partial\|^2_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} \\
&\qquad + 1_{\mathbb{R}}(a)\|f^a_\partial\|^2_{L^2(\mathbb{R},|p|dp;\mathfrak{L})},
\end{aligned}$$

and a uniform bound for $\|u_\varepsilon\|_{L^2(I,dq;\mathcal{H}^{1,L})}$.
As $\varepsilon \to 0$ the operators $(1+\varepsilon L_+)^{-1}$, (resp. $\mathrm{Id}_{\mathfrak{g}} \otimes (1+\varepsilon L_+)^{-1}$) converges in the strong operator topology to $\mathrm{Id}_{\mathfrak{L}}$ (resp. $\mathrm{Id}_{\mathfrak{g}\otimes\mathfrak{L}}$). Therefore we know

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \|f^{a,b}_{\partial,\varepsilon} - f^{a,b}_\partial\|_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} = 0, \\
&\lim_{\varepsilon\to 0} \|u_\varepsilon - u\|_{L^2(I,dq;\mathcal{H}^{1,0})} = 0, \\
&\lim_{\varepsilon\to 0} \|f_\varepsilon - f\|_{L^2(I,dq;\mathcal{H}^{-1,0})} = 0,
\end{aligned}$$

while u_ε converges weakly to some v in $L^2(I,dq;\mathcal{H}^{1,L})$. Since $\mathcal{H}^{-1,0} = (\mathcal{H}^{1,0})'$ is continuously embedded in $\mathcal{H}^{-1,L} = (\mathcal{H}^{1,L})'$, the second limit implies $v = u$ and

$$\|u\|^2_{L^2(I,dq;\mathcal{H}^{1,L})} \leq \liminf_{\varepsilon\to 0} \|u_\varepsilon\|^2_{L^2(I,dq;\mathcal{H}^{1,L})}.$$

This ends the proof. □

Remark 4.6. *Again the strong limit argument* $\lim_{\varepsilon\to 0}(1+\varepsilon L_+)^{-1}f = f$ *works in* $\mathfrak{L}$ *but not in* $D((\mathrm{Re}\ L)^{1/2})$, $\mathcal{H}^{1,L}$ *or* $\mathcal{H}^{-1,L}$.

4.3 Identifying the domains

We shall first consider the whole space problem on $\mathbb{R}^2$ and then the half-space problem in $\mathbb{R}_- \times \mathbb{R}$.
In the whole space problem, $(K^L_\pm, D(K^L_\pm))$ will denote the maximal accretive realization defined in $L^2(\mathbb{R}^2, dqdp;\mathfrak{L}) = L^2(\mathbb{R}^2, dqdp) \otimes \mathfrak{L}$ like in Subsection 3.3, with $\mathcal{H}_1 = L^2(\mathbb{R}^2, dqdp)$, $\mathcal{H}_2 = \mathfrak{L}$, $K_1 = K_\pm + \frac{1}{2} = \pm p\partial_q + (\frac{1}{2} + \mathcal{O})$ and $K_2 = L_\pm$ (the domains $D(K_\pm)$ and $D(L_\pm)$ are known).
For the half-space problem only the boundary conditions

$$\gamma_{odd} u = \pm\,\mathrm{sign}\,(p)\nu\gamma_{ev}u \quad \text{with} \quad \nu \in \{0,1\}$$

are considered. The corresponding maximal accretive realization of $\pm p.\partial_q + \frac{1}{2} + \mathcal{O}$ in $L^2(\mathbb{R}^2_-, dqdp)$ defined in Subsection 2.5 is denoted by $(K_{\pm,\nu} +$

$\frac{1}{2}, D(K_{\pm,\nu}))$. The maximal accretive operator $(K^L_{\pm,\nu}, D(K^L_{\pm,\nu}))$ is the maximal accretive realization defined in $L^2(\mathbb{R}^2_-, dqdp; \mathfrak{L}) = L^2(\mathbb{R}^2_-, dqdp) \otimes \mathfrak{L}$ like in Subsection 3.3, with $\mathcal{H}_1 = L^2(\mathbb{R}^2_-, dqdp)$, $\mathcal{H}_2 = \mathfrak{L}$, $K_1 = K_{\pm,\nu} + \frac{1}{2}$ and $K_2 = L_\pm$.

Proposition 4.7. *With Hypothesis 1, the domain $D(K^L_\pm)$ equals*

$$\left\{u \in L^2(\mathbb{R}, dq; \mathcal{H}^{1,L})\,,\ P^L_\pm u \in L^2(\mathbb{R}^2, dqdp; \mathfrak{L})\right\}\,.$$

The relation $K^L_\pm u = P^L_\pm u$ and the integration by part equality

$$\|u\|^2_{L^2(\mathbb{R},dq;\mathcal{H}^{1,L})} = \mathrm{Re}\,\langle u\,,\ K^L_\pm u\rangle$$

hold for any $u \in D(K^L_\pm)$.
The adjoint of $K^L_\pm$ is $K^L_\mp$.

Proof. From Theorem A.1 , we know that $K^*_\pm = K_\mp$.
The maximal accretive operator $K^L_\pm$ being defined as the generator of $(e^{-t(K_\pm+\frac{1}{2})}e^{-tL_\pm})_{t\geq 0}$, let us first check that $K^L_\pm u = P^L_\pm u$ when $u \in D(K^L_\pm)$. For such a $u \in D(K^L_\pm)$, $e^{-tK^L_\pm}u = e^{-t(K_\pm+\frac{1}{2})}e^{-tL_\pm}u$ defines an $L^2(\mathbb{R}^2, dqdp; \mathfrak{L})$-valued $\mathcal{C}^1$ function on $[0, +\infty)$. From the equality

$$(e^{-tK_\pm}e^{-tL_\pm})^* = e^{-tK^*_\pm}e^{-tL^*_\pm} = e^{-tK_\mp}e^{-tL_\mp}\,.$$

we deduce that $(K^L_\pm)^* = K^L_\mp$ and we know that $\mathcal{C}^\infty_0(\mathbb{R}^2) \otimes^{alg} D(L_-)$, and therefore $\mathcal{C}^\infty_0(\mathbb{R}^2; D(L_-))$, is contained in $D(K^L_\mp) = D((K^L_\pm)^*)$. Looking at the derivative at $t = 0$ of

$$\langle e^{-t(K_\mp+\frac{1}{2})}e^{-tL_\mp}v\,, u\rangle = \langle v\,,\ e^{-tK^L_\pm}u\rangle\,,$$

we obtain

$$\langle((K_\mp + \frac{1}{2}) \otimes \mathrm{Id} + \mathrm{Id} \otimes L_\mp)v\,,\ u\rangle = \langle v\,,\ K^L_\pm u\rangle\,,$$

for all $v \in \mathcal{C}^\infty_0(\mathbb{R}^2; D(L_\mp))$. But this means exactly

$$K^L_\pm u = (\frac{1}{2} + K_\pm)u + L_\pm u = P^L_\pm u \quad \text{in } \mathcal{D}'(\mathbb{R}^2; D(L_\mp)') \supset L^2(\mathbb{R}^2, dqdp; \mathfrak{L})\,.$$

Proposition 4.5 applied with $a = -\infty$, $b = +\infty$ and provides the inequality

$$\|u\|^2_{L^2(\mathbb{R},dq;\mathcal{H}^{1,L})} \leq \mathrm{Re}\,\langle u\,,\ K^L_\pm u\rangle \quad , \quad \forall u \in D(K^L_\pm)\,.$$

With $\|u\|_{L^2(\mathbb{R},dq;\mathcal{H}^{1,L})} \geq \|u\|$, this implies $\|K^L_\pm u\| \geq \|u\|_{L^2(\mathbb{R},dq;\mathcal{H}^{1,L})}$ and $D(K^L_\pm)$ is continuously embedded in $L^2(\mathbb{R}, dq; \mathcal{H}^{1,L})$.
Conversely, if v belongs to $L^2(\mathbb{R}, dq; \mathcal{H}^{1,L})$ with $P^L_\pm v = f \in L^2(\mathbb{R}^2, dqdp; \mathfrak{L})$, there exists $u \in D(K^L_\pm)$ such that $K^L_\pm u = f$ because $(K^L_\pm - 1, D(K^L_\pm))$ is maximal accretive. The difference $w = u - v$ belongs to $L^2(\mathbb{R}, dq; \mathcal{H}^{1,L})$ with $P^L w = 0$. Proposition 4.5 then implies

$$\|w\|^2_{L^2(\mathbb{R},dq;\mathcal{H}^{1,L})} \leq 0\,.$$

Therefore $v = u$ belongs to $D(K^L_\pm)$.
The equality

$$\mathrm{Re}\,\langle u\,,\,K^L_\pm u\rangle = \mathrm{Re}\,\langle u\,,\,(\frac{1}{2} + K_\pm)u\rangle + \mathrm{Re}\,\langle u\,,\,L_\pm u\rangle = \|u\|^2_{L^2(\mathbb{R},dq;\mathcal{H}^{1,L})}\,,$$

holds when $u \in D(K_\pm)\otimes^{alg} D(L_\pm)$. The algebraic tensor product $D(K_\pm)\otimes^{alg} D(L_\pm)$ is dense in $D(K^L_\pm)$ according to Proposition 3.11 while both sides are continuous on $D(K^L_\pm)$. This proves the equality for all $u \in D(K^L_\pm)$. □

Proposition 4.8. *Under Hypotheses 1 and with $\nu \in \{0,1\}$, the domain $D(K^L_{\pm,\nu})$ is nothing but*

$$\left\{u \in L^2(\mathbb{R}_-, dq; \mathcal{H}^{1,L})\,,\ \begin{array}{l} P^L_\pm u \in L^2(\mathbb{R}^2_-, dqdp; \mathfrak{L}) \\ \gamma_{odd} u = \pm\, sign\,(p)\nu\gamma_{ev} u \end{array}\right\}\,.$$

Moreover the relation

$$K^L_{\pm,\nu} u = P^L_\pm u\,,$$

and the integration by parts identity

$$\|u\|^2_{L^2(\mathbb{R}_-,dq;\mathcal{H}^{1,L})} + \|\gamma_{odd} u\|^2_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} = \mathrm{Re}\,\langle u\,,\,K^L_{\pm,\nu} u\rangle$$

holds for all $u \in D(K^L_{\pm,\nu})$.
Finally the adjoint of $K^L_{\pm,\nu}$ is $K^L_{\mp,\nu}$.

Remark 4.9. *The boundary condition $\gamma_{odd} u = \pm\, sign\,(p)\nu\gamma_{ev} u$ makes sense because the trace γu is well-defined in $L^2(\mathbb{R}, |p|dp; D(L_\mp)')$ when $u \in \mathcal{E}^{L,\pm}_{\mathbb{R}_-}$ and a fortiori when $u \in L^2(\mathbb{R}_-, dq; \mathcal{H}^{1,L})$ and $P^L_\pm u \in L^2(\mathbb{R}^2_-, dqdp; \mathfrak{L})$. For $\nu = 0$, the integration by parts inequality says nothing about $\gamma_{ev} u$. This will be studied later in Section 5.*

Proof. We proved in Proposition 2.3 that the adjoint of $K_{\pm,\nu}$ is $K_{\mp,\nu}$ for $\nu = \{0,1\}$.
The proof of $(K^L_{\pm,\nu})^* = K^L_{\mp,\nu}$ and $K^L_{\pm,\nu}u = P^L_{\pm}u$ when $u \in D(K^L_{\pm,\nu})$ is the same as in Proposition 4.7, with $\mathcal{C}^\infty_0(\mathbb{R}^2_-; D(L_\mp)) \subset D((K^L_{\pm,\nu})^*)$.
For $u \in D(K_{\pm,\nu}) \otimes^{alg} D(L_\pm)$ the integration by part formula

$$\|u\|^2_{L^2(\mathbb{R}_-,dq;\mathcal{H}^1\otimes\mathfrak{L})} + \mathbb{R}\mathrm{e}\,\langle \gamma_{ev}u\,,\,\nu\gamma_{ev}u\rangle_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} = \mathbb{R}\mathrm{e}\,\langle u\,,\,K^L_{\pm,\nu}u\rangle - \mathbb{R}\mathrm{e}\,\langle u\,,\,L_\pm u\rangle$$

comes from Theorem 2.1 or Proposition 2.10, applied with $\mathfrak{f}$ replaced by $\mathfrak{L}$ and

$$(\frac{1}{2} + K_{\pm,\nu})u = K^L_{\pm,\nu}u - L_\pm u \quad \text{in } L^2(\mathbb{R}^2_-, dqdp; \mathfrak{L})\,.$$

We know also, for such a u, $\gamma u \in L^2(\mathbb{R}, |p|dp; \mathfrak{L})$ and $\gamma_{odd}u = \pm\,\mathrm{sign}\,(p)\nu\gamma_{ev}u$.
With $\|u\|_{L^2(\mathbb{R}_-,dq;\mathcal{H}^1\otimes\mathfrak{L})} \geq \|u\|$, we deduce

$$\forall u \in D(K_{\pm,\nu})\otimes^{alg} D(L_\pm)\,, \quad \|u\|_{L^2(\mathbb{R}_-,dq;\mathcal{H}^{1,L})} + \|\gamma_{odd}u\|_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} \leq 8\|K^L_{\pm,\nu}u\|\,.$$

Since $D(K_{\pm,\nu}) \otimes^{alg} D(L_\pm)$ is dense in $D(K^L_{\pm,\nu})$, we conclude that for all $u \in D(K^L_{\pm,\nu})$, $\gamma_{odd}u = \pm\,\mathrm{sign}\,(p)\nu\gamma_{ev}u$ belongs to $L^2(\mathbb{R}, |p|dp; \mathfrak{L})$ and u belongs to $L^2(\mathbb{R}_-, dq; \mathcal{H}^{1,L})$.
With the same density argument, the equality

$$\|u\|^2_{L^2(\mathbb{R}_-,dq;\mathcal{H}^{1,L})} + \|\gamma_{odd}u\|^2_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} = \langle u\,,\,K^L_{\pm,\nu}u\rangle$$

can be extended to any $u \in D(K^L_{\pm,\nu})$.
Let us complete the identification of $D(K^L_{\pm,\nu})$. Since $K^L_{\pm,\nu} - 1$ is maximal accretive the equation $K^L_{\pm,\nu}u = f$ admits a unique solution $u \in D(K^L_{\pm,\nu})$ for any $f \in L^2(\mathbb{R}^2_-, dqdp; \mathfrak{L})$. If v belongs to $L^2(\mathbb{R}_-, dq; \mathcal{H}^{1,L})$ and satisfies

$$\begin{aligned}
&P^L_\pm v = f \in L^2(\mathbb{R}^2_-, dqdp; \mathfrak{L})\,,\\
&\gamma_{odd}v = \pm\,\mathrm{sign}\,(p)\nu\gamma_{ev}v \quad (\text{in } L^2(\mathbb{R}, |p|dp; D(L_\mp)')\,,
\end{aligned}$$

solve $K^L_{\pm,\nu}u = f$. The difference $w = u - v$ belongs to $L^2(\mathbb{R}_-, dq; \mathcal{H}^{1,L})$ and satisfies

$$P^L_\pm w = 0 \quad , \quad \gamma_{odd}w = \pm\,\mathrm{sign}\,(p)\nu\gamma_{ev}w\,.$$

This implies $w \in \mathcal{E}^{L,\pm}_{\mathbb{R}_-} \subset \mathcal{E}_{\mathbb{R}_-}(D(L_\mp)')$ according to Proposition 4.2. The regularisation $w' = (1+L)^{-1}w$ belongs to $\mathcal{E}_{\mathbb{R}_-}(\mathfrak{L})$ and has a trace $\gamma w' \in L^2(\mathbb{R}, |p|dp; \mathfrak{L})$. It satisfies $P^L_\pm w' = 0$, $w' \in L^2(\mathbb{R}_-; \mathcal{H}^{1,L)})$ and $\gamma w' \in$

$L^2(\mathbb{R}, |p|dp; \mathfrak{L})$ with $\gamma_{odd}w' = \pm\,\mathrm{sign}\,(p)\nu\gamma_{ev}w'$. The integration by part inequality of Proposition 4.5 applies to w':

$$\|w'\|^2_{L^2(\mathbb{R}_-;\mathcal{H}^{1,L})} + \nu\langle\gamma_{ev}w'\,,\gamma_{ev}w'\rangle \leq 0\,,$$

which yields $w' = 0$, $w = 0$ and $v = u \in D(K^L_{\pm,\nu})$. □

Proposition 4.10. *Assume Hypotheses 1 and take* $\lambda \in \mathbb{R}$.

- *The resolvent* $(K^L_\pm - i\lambda)^{-1}$ *admits a continuous extension which sends* $L^2(\mathbb{R}, dq; \mathcal{H}^{-1,L})$ *into* $L^2(\mathbb{R}, dq; \mathcal{H}^{1,L})$ *and the equation*

$$(P^L_\pm - i\lambda)u = f$$

has a unique solution $u \in L^2(\mathbb{R}, dq; \mathcal{H}^{1,L})$ *when* $f \in L^2(\mathbb{R}, dq; \mathcal{H}^{-1,L})$.

- *For* $\nu \in \{0, 1\}$, *the resolvent* $(K^L_{\pm,\nu} - i\lambda)^{-1}$ *admits a continuous extension which sends* $L^2(\mathbb{R}_-, dq; \mathcal{H}^{-1,L})$ *into* $L^2(\mathbb{R}_-, dq; \mathcal{H}^{1,L})$ *and the equation*

$$(P^L_\pm - i\lambda)u = f \quad , \quad \gamma_{odd}u = \pm\,\mathit{sign}\,(p)\nu\gamma_{ev}u\,,$$

has a unique solution $u \in L^2(\mathbb{R}_-, dq; \mathcal{H}^{1,L})$ *when* $f \in L^2(\mathbb{R}_-, dq; \mathcal{H}^{-1,L})$, *with*

$$\|u\|^2_{L^2(\mathbb{R}_-,dq;\mathcal{H}^{1,L})} + \|\gamma_{odd}u\|^2_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} = \mathbb{R}\mathrm{e}\,\langle f\,,\,u\rangle\,.$$

Proof. The proof is the same in the two cases and even simpler when there is no boundary term in the first case. We focus on the second case.
For $f \in L^2(\mathbb{R}^2_-, dq; \mathfrak{L})$ and $u = (K^L_{\pm,\nu} - i\lambda)^{-1}f \in D(K^L_{\pm,\nu})$, the inequalities

$$\begin{aligned}\|u\|^2_{L^2(\mathbb{R}_-,dq;\mathcal{H}^{1,L})} + \|\gamma_{odd}u\|^2_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} &= \mathbb{R}\mathrm{e}\,\langle u\,,\,(K^L_{\pm,\nu} - i\lambda)u\rangle \\ &= \mathbb{R}\mathrm{e}\,\langle u\,,\,f\rangle \\ &\leq \|u\|_{L^2(\mathbb{R}_-,dq;\mathcal{H}^{1,L})}\|f\|_{L^2(\mathbb{R}_-,dq;\mathcal{H}^{-1,L})}\,,\end{aligned}$$

allows to extend $(K^L_{\mp,\nu} - i\lambda)^{-1}f$ to $f \in L^2(\mathbb{R}_-, dq; \mathcal{H}^{-1,L})$. Approximating a general $f \in L^2(\mathbb{R}_-, dq; \mathcal{H}^{-1,L})$ by $f_n \in L^2(\mathbb{R}^2_-; \mathfrak{L})$ implies that the above equalities are still valid for $f \in L^2(\mathbb{R}_-, dq; \mathcal{H}^{-1,L})$.
Assume now that $v \in L^2(\mathbb{R}_-, dq; \mathcal{H}^{1,L})$ solves $(P^L_\pm - i\lambda)v = f \in L^2(\mathbb{R}_-, dq; \mathcal{H}^{-1,L})$ and $\gamma_{odd}v = \pm\,\mathrm{sign}\,(p)\nu\gamma_{ev}v$ in $L^2(\mathbb{R}, |p|dp; D(L_\mp)')$. Then the difference $w = v - (K^L_{\pm,\nu} - i\lambda)^{-1}f$ belongs to $L^2(\mathbb{R}_-; \mathcal{H}^{1,L})$ and satisfies $P^L w = i\lambda w \in L^2(\mathbb{R}^2_-, dqdp; \mathfrak{L})$, $\gamma_{odd}w = \pm\,\mathrm{sign}\,(p)\nu\gamma_{ev}w$. It belongs to $D(K^L_{\pm,\nu})$ and solves $(K^L_{\pm,\nu} - i\lambda)w = 0$. This implies $w = 0$ and the solution is unique. □

Definition 4.11. *We keep the notations* $(K^L_\pm - i\lambda)^{-1}f$ *when* $f \in L^2(\mathbb{R}, dq; \mathcal{H}^{-1,L})$ *and* $(K^L_{\pm,\nu} - i\lambda)^{-1}f$ *when* $f \in L^2(\mathbb{R}_-, dq; \mathcal{H}^{-1,L})$.

4.4 Inhomogeneous boundary value problems

In this paragraph, we study an inhomogeneous boundary value problems, naturally associated with $K^L_{+,1}$ and $K^L_{+,0}$. The control of $\gamma_{ev}u$ for $K^L_{\pm,1}$ allows the variational arguments of [Luc][Car][Lio]. A general a priori estimate is deduced.

Proposition 4.12. *Assume Hypothesis 1, $f_\partial = \Pi_\mp L^2(\mathbb{R}, |p|dp; \mathfrak{L}) = Ran\,\Pi_{ev}$, $f \in L^2(\mathbb{R}_-, dq; \mathcal{H}^{-1,L})$ and $\lambda \in \mathbb{R}$. Then the boundary value problem*

$$(P^L_\pm - i\lambda)u = f \quad , \quad \Pi_\mp \gamma u = \gamma_{ev}u \mp sign\,(p)\gamma_{odd}u = f_\partial\,, \tag{48}$$

admits a unique solution in $L^2(\mathbb{R}_-, dq; \mathcal{H}^{1,L})$, which is characterized by

$$\forall \varphi \in D(K^L_{-,1})\,, \quad \langle u\,,\,(K^L_{-,1} + i\lambda)\varphi\rangle = \langle u\,,\,f\rangle + \frac{1}{2}\langle f_\partial\,,\,\Pi_\pm\gamma\varphi\rangle_{L^2(\mathbb{R},|pdp|;\mathfrak{L})}\,. \tag{49}$$

It satisfies the integration by part identity

$$\frac{1}{4}\|\Pi_\pm\gamma u\|^2_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} + \|u\|^2_{L^2(\mathbb{R}_-,dq;\mathcal{H}^{1,L})} = \frac{1}{4}\|f_\partial\|^2_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} + \mathrm{Re}\,\langle f\,,\,u\rangle\,. \tag{50}$$

Proof. In this proof it is important to distinguish P^L_+ and P^L_-. While focusing on P^L_+ (the other case is deduced by a transposition of $\{+,-\}$), we will work with P^L_+ and P^L_-.

Uniqueness: If there are two solutions $u_1, u_2 \in L^2(\mathbb{R}_-, dq; \mathcal{H}^{1,L})$ then the difference w belongs to $\mathcal{E}^{L,+}_{\mathbb{R}_-}$ and solves $(P^L_+ - i\lambda)w = 0$ with $\gamma_{odd}w = \mathrm{sign}\,(p)\gamma_{ev}w \in L^2(\mathbb{R}, |p|dp; D(L_-)')$. This means exactly $w = (K^L_{+,1} - i\lambda)^{-1}0 = 0$ and $u_2 = u_1$.

Existence when $f_\partial = 0$**:** Simply apply Proposition 4.10 and $u = (K^L_{+,1} - i\lambda)^{-1}f$ according to Definition 4.11. Because $(K^L_{+,1}) = K^L_{-,1}$, the solution is characterized by

$$\forall \varphi \in D(K^L_{-,1})\,, \quad \langle u\,,\,(K^L_{-,1} + i\lambda)\varphi\rangle = \langle f\,,\,\varphi\rangle\,.$$

Existence when $f = 0$**:** We adapt the approach of [Luc][Car]. Let V be the space $L^2(\mathbb{R}_-, dq; \mathcal{H}^{1,L})$ and let Y be the domain $D(K^L_{-,1})$ in $L^2(\mathbb{R}^2_-, dqdp; \mathfrak{L})$: We are in the case $\nu = +1$ and Y is the set of $\varphi \in L^2(\mathbb{R}_-, dq; \mathcal{H}^{1,L})$ which satisfy

$$\begin{aligned}&(P^L_- + i\lambda)\varphi \in L^2(\mathbb{R}^2_-, dqdp; \mathfrak{L})\,,\\ &\gamma\varphi \in L^2(\mathbb{R}, |p|dp; \mathfrak{L}) \quad , \quad \Pi_+\gamma\varphi = 0 \quad (\text{or} \quad \gamma_{ev}\varphi + \mathrm{sign}\,(p)\gamma_{odd}\varphi = 0)\,.\end{aligned}$$

The space Y is endowed with the norm

$$\|\varphi\|_Y = \left(\|\varphi\|^2_{L^2(\mathbb{R}_-,dq;\mathcal{H}^{1,L})} + \|\gamma_{ev}\varphi\|^2_{L^2(\mathbb{R},|p|dp;\mathfrak{L})}\right)^{\frac{1}{2}} .$$

The space $(Y, \|\ \|_Y)$ is continuously embedded in $V = L^2(\mathbb{R}_-, dq; \mathcal{H}^{1,L})$ and contains $\left\{\varphi \in \mathcal{C}_0^\infty(\overline{\mathbb{R}^2_-}; D(L_-))\,, 1_{\mathbb{R}_+}(p)\varphi(0,p) \equiv 0\right\}$, which is dense in V. Let $a : V \times Y \to \mathbb{R}$ be the real bilinear form given by

$$a(u, \varphi) = \mathbb{R}\mathrm{e}\,\langle u\,,\,(P^L_- + i\lambda)\varphi\rangle\,,$$

For any fixed $\varphi \in Y$ the real linear map $V \ni u \mapsto a(u, \varphi) \in \mathbb{R}$ is continuous. For $\varphi \in Y = D(K^L_{-,1})$, Proposition 4.8 implies that $a(\varphi, \varphi)$ equals

$$a(\varphi,\varphi) \;=\; \mathbb{R}\mathrm{e}\,\langle\varphi\,,\,(K^L_{-,1} + i\lambda)\varphi\rangle \;=\; \|\gamma_{ev}\varphi\|^2_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} + \|\varphi\|^2_{L^2(\mathbb{R}_-,dq;\mathcal{H}^{1,L})}$$

and a is coercive on $(Y, \|\ \|_Y)$. Finally the linear form $\ell : F \to \mathbb{R}$ given by

$$\ell(\varphi) = -\,\mathbb{R}\mathrm{e}\,\langle f_\partial\,,\,\mathrm{sign}\,(p)\gamma\varphi\rangle_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} = \frac{1}{2}\,\mathbb{R}\mathrm{e}\,\langle \Pi_- f_\partial\,,\,\Pi_-\gamma\varphi\rangle_{L^2(\mathbb{R},|p|dp;\mathfrak{L})}\,,$$

where we use (46) and $\Pi_+\gamma\varphi = 0$, is continuous.
By Lions'Theorem (see [Lio]) there exists $u \in V$ such that

$$\forall \varphi \in Y\,,\quad a(u,\varphi) = \frac{1}{2}\,\mathbb{R}\mathrm{e}\,\langle \Pi_- f_\partial\,,\,\Pi_-\gamma\varphi\rangle_{L^2(\mathbb{R},|p|dp;\mathfrak{L})}\,.$$

By taking test functions $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2_-; D(L_-)) \subset Y$ supported away from $\{q = 0\}$, the integration by part of Proposition 4.3 (first case) implies

$$(P^L_+ - i\lambda)u = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^2_-; D(L_-)') \supset L^2(\mathbb{R}^2_-, dqdp; \mathfrak{L})\,.$$

Now u and any $\varphi \in Y$ fulfill the assumptions of the second case of Proposition 4.3. We obtain

$$\begin{aligned}
\frac{1}{2}\,\mathbb{R}\mathrm{e}\,\langle \Pi_- f_\partial\,,\,\Pi_-\varphi\rangle_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} \quad &= a(u,\varphi)\\
&= 0 - \lim_{\varepsilon\to 0^+}\mathbb{R}\mathrm{e}\,\langle (1+\varepsilon L_+)^{-1}\gamma u\,,\,\mathrm{sign}\,(p)(1+\varepsilon L_-)^{-1}\gamma\varphi\rangle_{L^2(\mathbb{R},|p|dp;\mathfrak{L})}\,,\\
&= \lim_{\varepsilon\to 0^+}\frac{1}{2}\,\mathbb{R}\mathrm{e}\,\langle (1+\varepsilon L_+)^{-1}\Pi_-\gamma u\,,\,(1+\varepsilon L_-)^{-1}\Pi_-\gamma\varphi\rangle_{L^2(\mathbb{R},|p|dp;\mathfrak{L})}\,.
\end{aligned}$$

By taking an arbitrary $\gamma\varphi \in \mathcal{C}_0^\infty((-\infty,0);D(L_-))$, the right-hand side converges to

$$\frac{1}{2}\langle \Pi_-\gamma u\,,\,\Pi_-\gamma\varphi\rangle_{L^2(\mathbb{R},|p|dp;D(L_-)'),L^2(\mathbb{R},|p|dp;D(L_-))}\,.$$

Since the equality holds for all

$$\varphi \in \Pi_-\mathcal{C}_0^\infty((-\infty,0);D(L_-)) = \Pi_-\mathcal{C}_0^\infty(\mathbb{R}^*;D(L_-)) = \Pi_{ev}\mathcal{C}_0^\infty(\mathbb{R}^*;D(L_-))\,,$$

this proves

$$\Pi_-\gamma u = \Pi_- f_\partial = f_\partial\,.$$

The solution u is characterized by

$$\forall\varphi \in D(K_{-,1}^L)\,,\quad \langle u\,,\,(K_{-,1}^L + i\lambda)\varphi\rangle = \frac{1}{2}\langle f_\partial\,,\,\Pi_-\gamma\varphi\rangle_{L^2(\mathbb{R},|pdp|;\mathfrak{L})}\,,$$

and (49) is obtained by adding the two cases $f=0$ and $f_\partial = 0$.
Estimate: We prove now that the solution to (48) satisfies

$$\|u\|_{L^2(\mathbb{R}_-,dq;\mathcal{H}^{1,L})} + \|\gamma u\|_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} \leq C\left[\|f\|_{L^2(\mathbb{R}_-,dq;\mathcal{H}^{-1,L})} + \|f_\partial\|_{L^2(\mathbb{R},|p|dp;\mathfrak{L})}\right]\,, \tag{51}$$

where $C>0$ is independent of $\lambda\in\mathbb{R}$. The solution u to (48) is the sum

$$u = u_1 + u_0 = (K_{+,1}^L - i\lambda)^{-1}f + u_0\,,$$

where u_0 solves the boundary value problem (48) with $f=0$.
Since $f\in L^2(\mathbb{R}_-,dq;\mathcal{H}^{-1,L})$, Proposition 4.10 gives

$$\begin{aligned}\|u_1\|^2_{L^2(\mathbb{R}_-,dq;\mathcal{H}^{1,L})} + \|\gamma_{odd}u_1\|^2_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} &= \mathrm{Re}\,\langle u_1\,,\,f\,\rangle_{L^2(\mathbb{R}_-,dq;\mathcal{H}^{1,L})\,,L^2(\mathbb{R}_-,dq;\mathcal{H}^{-1,L})}\\ &\leq \|u_1\|_{L^2(\mathbb{R}_-,dq;\mathcal{H}^{1,L})}\|f\|_{L^2(\mathbb{R}_-,dq;\mathcal{H}^{-1,L})}\,,\end{aligned}$$

where the boundary condition $\gamma_{odd}u_1 = \mathrm{sign}\,(p)\gamma_{ev}u_1$ implies

$$\|\gamma_{odd}u_1\|^2_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} = \frac{1}{2}\|\gamma u_1\|^2_{L^2(\mathbb{R},|p|dp;\mathfrak{L})}\,.$$

We deduce

$$\|u_1\|_{L^2(\mathbb{R}_-,dq;\mathcal{H}^{1,L})} + \|\gamma u\|_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} \leq 2\|f\|_{L^2(\mathbb{R},dq;\mathcal{H}^{-1,L})}\,.$$

The function $u_0 \in L^2(\mathbb{R}_-,dq;\mathcal{H}^{1,L})$ solves (48) with $f=0$. Therefore, it belongs to $\mathcal{E}_{\mathbb{R}_-}^{L,+}$ and has a trace $\gamma u \in L^2(\mathbb{R},|p|dp;D(L_-)')$ according to

Proposition 4.2. We follow the regularization scheme of Proposition 4.5. For $\varepsilon > 0$ the function $u_\varepsilon = (1+\varepsilon L_+)^{-1}u_0$ belongs to $L^2(\mathbb{R}_-, dq; \mathcal{H}^{1,L})$ and satisfies

$$(P_+^L - i\lambda)u_\varepsilon = 0 \quad , \quad \gamma u_\varepsilon \in L^2(\mathbb{R}, |p|dp; \mathfrak{L}) .$$

Hence the integration by part inequality of Proposition 4.5 applies and gives

$$\begin{aligned}
2\|u_\varepsilon\|^2_{L^2(\mathbb{R}_-,dq;\mathcal{H}^{1,L})} &\leq -\int_{\mathbb{R}} |\gamma u_\varepsilon(p)|^2_{\mathfrak{L}}\, pdp \\
&\leq -2\,\mathbb{R}\mathrm{e}\,\langle \gamma_{ev}u_\varepsilon\,,\, \mathrm{sign}\,(p)\gamma_{odd}u_\varepsilon\rangle_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} \\
&\leq -\frac{1}{2}\|\Pi_+\gamma u_\varepsilon\|^2_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} + \frac{1}{2}\|f_\partial\|^2_{L^2(\mathbb{R},|p|dp;\mathfrak{L})}\,,
\end{aligned}$$

where (46) and $\|(1+\varepsilon L_+)^{-1}f_\partial\|_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} \leq \|f\|_{L^2(\mathbb{R},|p|dp;\mathfrak{L})}$ were used for the last inequality.
With

$$\begin{aligned}
\|\gamma u_\varepsilon\|_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} &\leq \frac{\|\Pi_+\gamma u_\varepsilon\|_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} + \|\Pi_-\gamma u_\varepsilon\|_{L^2(\mathbb{R},|p|dp;\mathfrak{L})}}{2} \\
&\leq \|f_\partial\|_{L^2(\mathbb{R},|p|dp;\mathfrak{L})}\,,
\end{aligned}$$

we infer the uniform bound

$$\|u_\varepsilon\|_{L^2(\mathbb{R}_-,dq;\mathcal{H}^{1,L})} + \|\gamma u_\varepsilon\|_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} \leq \frac{3}{2}\|f_\partial\|_{L^2(\mathbb{R},|p|dp;\mathfrak{L})}\,,$$

while we know the strong convergences $\lim_{\varepsilon\to 0} u_\varepsilon = u_0$ in $L^2(\mathbb{R}^2_-, dqdp; \mathfrak{L})$ and $\lim_{\varepsilon\to 0}\gamma u_\varepsilon = \gamma u_0$ in $L^2(\mathbb{R}, |p|dp; D(L_-)')$. Therefore $u_0 \in L^2(\mathbb{R}_-, dq; \mathcal{H}^{1,L})$ has a trace $\gamma u_0 \in L^2(\mathbb{R}, |p|dp; \mathfrak{L})$ and

$$\begin{aligned}
\|u_0\|_{L^2(\mathbb{R}_-,dq;\mathcal{H}^{1,L})} + \|\gamma u_0\|_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} &\leq \liminf_{\varepsilon\to 0^+} \big[\|u_\varepsilon\|_{L^2(\mathbb{R}_-,dq;\mathcal{H}^{1,L})} \\
&\qquad\qquad + \|\gamma u_\varepsilon\|_{L^2(\mathbb{R},|p|dp;\mathfrak{L})}\big] \\
&\leq \frac{3}{2}\|f_\partial\|_{L^2(\mathbb{R},|p|dp;\mathfrak{L})}\,.
\end{aligned}$$

Integration by part identity: By (51) all the terms of (50) are continuous with respect to $f \in L^2(\mathbb{R}_-, dq; \mathcal{H}^{-1,L})$. We can therefore assume $f \in L^2(\mathbb{R}^2_-, dqdp; \mathfrak{L})$. We set $f_\varepsilon = (1+\varepsilon L)^{-1}f$ and $f_{\partial,\varepsilon} = (1+\varepsilon L)^{-1}f_\partial$. Then $u_\varepsilon = (1+\varepsilon L)u \in L^2(\mathbb{R}_-, dq; \mathcal{H}^{1,L})$ is the unique solution to

$$(P_+^L - i\lambda)u_\varepsilon = f_\varepsilon \quad , \quad \Pi_-\gamma u_\varepsilon = f_{\partial,\varepsilon}\,.$$

This implies

$$p\partial_q u_\varepsilon = f_\varepsilon + i\lambda u_\varepsilon - (\frac{1}{2} + \mathcal{O})u_\varepsilon - Lu_\varepsilon \quad \in L^2(\mathbb{R}_-, dq; \mathcal{H}^{-1} \otimes \mathfrak{L})\,,$$

while

$$u_\varepsilon \in L^2(\mathbb{R}_-, dq; \mathcal{H}^1 \otimes \mathfrak{L}) \quad \text{and} \quad \gamma u_\varepsilon = (1+\varepsilon L)^{-1}\gamma u \in L^2(\mathbb{R}, |p|dp; \mathfrak{L})\,.$$

Proposition 2.10 applied with $\mathfrak{f} = \mathfrak{L}$, $a = -\infty$ and $b = 0$, says

$$\begin{aligned}2\,\mathbb{R}\mathrm{e}\,\langle u_\varepsilon\,, p\partial_q u_\varepsilon\rangle &= \int_{\mathbb{R}} |u(0,p)|^2_{\mathfrak{L}}\; pdp = 2\,\mathbb{R}\mathrm{e}\,\langle \gamma_{ev} u_\varepsilon\,, \mathrm{sign}\,(p)\gamma_{odd} u_\varepsilon\rangle_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} \\ &= \frac{1}{2}\left[\|\Pi_+\gamma u_\varepsilon\|^2_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} - \|\Pi_-\gamma u_\varepsilon\|^2_{L^2(\mathbb{R},|p|dp;\mathfrak{L})}\right]\,.\end{aligned}$$

We obtain

$$\mathbb{R}\mathrm{e}\,\langle u_\varepsilon\,,\, f_\varepsilon\rangle + \frac{1}{4}\|f_{\partial,\varepsilon}\|^2_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} = \frac{1}{4}\|\Pi_+\gamma u_\varepsilon\|^2_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} + \|u_\varepsilon\|^2_{L^2(\mathbb{R}_-,dq,\mathcal{H}^{1,L})}\,.$$

With $f \in L^2(\mathbb{R}^2_-, dqdp; \mathfrak{L})$, taking the limit as $\varepsilon \to 0$ gives

$$\begin{aligned}&\lim_{\varepsilon}\|f - f_\varepsilon\|_{L^2(\mathbb{R}_-,dq;\mathcal{H}^{-1,L})} \leq \lim_{\varepsilon\to 0}\|f - f_\varepsilon\| = 0\,,\\ &\lim_{\varepsilon\to 0}\|f_\partial - f_{\partial,\varepsilon}\|_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} = 0\,.\end{aligned}$$

Owing to the continuity estimate (51) (this is the important point), we deduce

$$\begin{aligned}&\lim_{\varepsilon}\|u - u_\varepsilon\|_{L^2(\mathbb{R}_-,dq;\mathcal{H}^{1,L})} = 0\,,\\ &\lim_{\varepsilon\to 0}\|\gamma u - \gamma u_\varepsilon\|_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} = 0\,,\end{aligned}$$

and (50) is proved. □

An easy consequence of Proposition 4.12 is the following result.

Proposition 4.13. *If $u \in L^2(\mathbb{R}_-, dq; \mathcal{H}^{1,L})$ solves $(P_\pm - i\lambda)u = 0$ with $\gamma u \in L^2(\mathbb{R}, |p|dp; \mathfrak{L})$, then*

$$\pm 2\,\mathbb{R}\mathrm{e}\,\langle \gamma_{ev} u\,,\, sign\,(p)\gamma_{odd} u\rangle = -\|u\|^2_{L^2(\mathbb{R}_-,dq;\mathcal{H})^1} \leq 0\,.$$

Proof. The function u solves (48) with $f_\partial = \Pi_\mp \gamma u$. Apply simply (50) by referring again to (46). □

Definition 4.14. *With the inhomogeneous boundary value problem* (48) *with* $f = 0$, $\lambda \in \mathbb{R}$, $f_\partial \in \Pi_{ev}L^2(\mathbb{R}, |p|dp; \mathfrak{L})$ *and* $u \in L^2(\mathbb{R}_-, dq; \mathcal{H}^{1,L})$ *solution, we define the two operators*

$$R_\pm^L(\lambda) : \Pi_{ev}L^2(\mathbb{R}, |p|dp; \mathfrak{L}) \to L^2(\mathbb{R}_-, dq; \mathcal{H}^{1,L}) \quad , \quad R_\pm^L(\lambda)f_\partial = u \,,$$
$$C_\pm^L(\lambda) : \Pi_{ev}L^2(\mathbb{R}, |p|dp; \mathfrak{L}) \to \Pi_{ev}L^2(\mathbb{R}, |p|dp; \mathfrak{L}) \quad , \quad C_\pm^L f_\partial = \Pi_\pm \gamma u \,,$$

with $C_\pm^L(\lambda) = \Pi_\pm \circ \gamma \circ R_\pm^L(\lambda)$.

Proposition 4.15. *Assume Hypothesis 1 and take* $\lambda \in \mathbb{R}$ *.*
The operator $R_\pm^L(\lambda)$ *is continuous and injective.*
Its adjoint is

$$(R_\pm^L(\lambda))^* = \Pi_\mp \circ \gamma \circ (K_\mp^L + i\lambda)^{-1} : L^2(\mathbb{R}_-, dq; \mathcal{H}^{-1,L}) \to \Pi_{ev}L^2(\mathbb{R}, |p|dp; \mathfrak{L})$$

and its range is dense in $\Pi_{ev}L^2(\mathbb{R}, |p|dp; \mathfrak{L})$*.*
The operator $C_\pm^L(\lambda)$ *is a contraction of* $\Pi_{ev}L^2(\mathbb{R}, dp; \mathfrak{L})$*:*

$$\forall f_\partial \in \Pi_{ev}L^2(\mathbb{R}, |p|dp; \mathfrak{L}) \,, \quad \|C_\pm^L(\lambda)f_\partial\|_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} \leq \|f_\partial\|_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} \,.$$

Proof. The operators $R_\pm(\lambda)$ and $C_\pm(\lambda)$ are continuous owing to (50). This also proves that $C_\pm(\lambda)$ is a contraction.
Injectivity: If $u = R_\pm^L(\lambda)f_\partial = 0$ then $\gamma u = 0$ and $f_\partial = \Pi_\mp \gamma u = 0$.
Adjoint: In the proof of Proposition 4.12, $u = R_\pm^L f_\partial$ is characterized by

$$\forall \varphi \in D(K_{\mp,1}^L) \,, \quad \langle u \,, \, (K_{\mp,1}^L + i\lambda)\varphi \rangle = \frac{1}{2} \langle f_\partial \,, \, \Pi_\mp \varphi \rangle_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} \,.$$

But both sides extend by continuity to any φ such that $(K_\mp^L + i\lambda)\varphi \in L^2(\mathbb{R}_-, dq; \mathcal{H}^{-1,L})$ according to Proposition 4.10. Hence we get

$$\forall f \in L^2(\mathbb{R}_-, dq; \mathcal{H}^{-1,L}) \,, \quad \langle R_\pm^L f_\partial \,, \, f \rangle = \frac{1}{2} \langle f_\partial \,, \, \Pi_\mp \gamma (K_\mp^L + i\lambda)^{-1} f \rangle_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} \,.$$

This proves $R_\pm^L(\lambda)^* = \frac{1}{2}\Pi_\mp \circ \gamma \circ (K^L + i\lambda)^{-1}$ and since $R_\pm^L$ is injective, $R_\pm^L(\lambda)^*$ has a dense range. $\square$

One may wonder about $C_\pm(\lambda)$ being a strict contraction:

$$\|C_\pm^L(\lambda)\|_{\mathcal{L}(\Pi_{ev}L^2(\mathbb{R},|p|dp;\mathfrak{L}))} < 1 \,.$$

The following statements are equivalent:

i) $R^L_\pm(\lambda)$ has a closed range;

ii) $R^L_\pm(\lambda)^*$ is surjective;

iii) $C^L_\pm(\lambda)$ is a strict contraction;

Proving one of them, for the specific case when $L_\pm$ is a geometric Kramers-Fokker-Planck operator, would allow to consider more general boundary conditions and to relax the gap assumption $c_A > 0$ in (7) for Theorem 1.2.

5 General boundary conditions for half-space problems

The analysis of abstract half-space problems is enhanced by introducing more general boundary conditions which possibly couple the variables. We keep the notations introduced in Section 4.
By referring to Appendix A, the analysis developed here proves Theorem 1.1 and Theorem 1.2 when $\overline{Q} = (-\infty, 0] \times \mathbb{T}^{d_1'} \times \mathbb{R}^{d_2'}$ is endowed with the euclidean metric.

5.1 Assumptions for L and A

We keep the notations introduced in Subsection 4.1. We shall need some additional assumptions for the vertical maximal accretive operators $(L_\pm, D(L_\pm))$, $L_- = L_+^*$, defined in $\mathfrak{L}$. Those are formulated in terms of the maximal accretive operators $K^L_\pm = P^L_\pm = \pm p\partial_q + \frac{1}{2} + \mathcal{O} + L_\pm$, defined on the whole line $q \in \mathbb{R}$, with $D(K^L_\pm) \subset L^2(\mathbb{R}^2, dqdp; \mathfrak{L})$. We have checked in Proposition 4.7 the equality $(K^L_\pm)^* = K^L_\mp$.
Remember that Hypothesis 1 says that $D(L_+) \cap D(L_-)$ is a core for $\mathbb{R}\mathrm{e}\,(L)^{1/2}$. This is strengthened by

Hypothesis 2. *The domain $D(L_+) = D(L_-)$ is a core for $\mathbb{R}\mathrm{e}\,(L)$. The domains $D(K^L_\pm)$ and $D(K^L_\mp)$ for the whole line problem are equal, and the two operators K^L_+ and K^L_- fulfill the estimate*

$$\|(K^L_\pm - K^L_\mp - 2i\lambda)u\| + \|(K^L_\pm + K^L_\mp)u\| + \langle\lambda\rangle^{1/2}\|u\| + \|u\|_{\mathcal{Q}_0} \leq C_L \|(K^L_\pm - i\lambda)u\|\,,$$

for all $\lambda \in \mathbb{R}$ and all $u \in D(K^L_\pm)$ and where $\mathcal{Q}_0$ is a Hilbert space embedded in $L^2(\mathbb{R}^2, dqdp; \mathfrak{L})$.

In applications, the Hilbert space $\mathcal{Q}_0$ will encode the regularity properties with respect to q. A stronger version is also needed for general boundary conditions.

Hypothesis 3. *There exists a Hilbert space $\mathcal{Q}$ embedded in $L^2(\mathbb{R}^2, dqdp; \mathfrak{L})$ such that any solution $v \in L^2(\mathbb{R}, dq; \mathcal{H}^{1,L})$ to*

$$(P^L_\pm - i\lambda)v = \gamma\delta_0(q) \quad \textit{in } \mathcal{S}'(\mathbb{R}^2; D(L_\mp)')\,, \tag{52}$$

with $\gamma \in L^2(\mathbb{R}, \frac{dp}{|p|}; \mathfrak{L})$, satisfies

$$\|v\|_{\mathcal{Q}} \leq C_{\mathcal{Q},L}\left[\|\gamma\|_{L^2(\mathbb{R},\frac{dp}{|p|};\mathfrak{L})} + \|v\|_{L^2(\mathbb{R},dq;\mathcal{H}^{1,L})}\right]. \tag{53}$$

We also assume

$$\forall u \in D(K^L_\pm)\,, \quad \langle\lambda\rangle^{\frac{1}{4}}\|u\|_{\mathcal{Q}} \leq C_{\mathcal{Q},L}\|(K^L_\pm - i\lambda)u\|\,. \tag{54}$$

Example: An easy example which fulfills all these assumptions is the Kramers-Fokker-Planck operator $L_\pm = \pm p'.\partial_{q'} + \frac{-\Delta_{p'}+|p'|^2}{2}$ defined in $L^2(X', dq'dp')$ with $X' = T^*Q'$ and $Q' = \mathbb{R}^{d'_1} \times \mathbb{T}^{d'_2}$ endowed in the euclidean metric. The operator $K^L_\pm - \frac{1}{2} = p.\partial_q + \frac{-\Delta_p+|p|^2}{2}$ is then the Kramers-Fokker-Planck operator in $L^2(T^*Q, dqdpdq'dp')$ and $Q = \mathbb{R}^{d'_1+1} \times \mathbb{T}^{d'_2}$ is endowed with the euclidean metric. Hypothesis 2 is thus verified with $\mathcal{Q}_0 = H^{\frac{2}{3}}(Q; \mathcal{H}^0) = H^{\frac{2}{3}}(Q; L^2(\mathbb{R}^{1+d'_1+d'_2}, dp))$ according to Theorem A.1 applied with $s = s' = 0$. Hypothesis 3 is verified with $\mathcal{Q} = H^t(Q; \mathcal{H}^0) = H^t(Q; L^2(\mathbb{R}^{1+d'_1+d'_2}, dp))$ for any $t \in [0, \frac{1}{9})$: the first statement and the estimate (52) is provided by Proposition A.5; the estimate (54) is inferred by interpolating the inequalities of Theorem A.1 with $s = s' = 0$,

$$\|u\|^2_{H^t(Q;\mathcal{H}^0)} \leq \|u\|^2_{H^{\frac{1}{3}}(Q;\mathcal{H}^0)} \leq C\|u\|_{H^{\frac{2}{3}}(Q;\mathcal{H}^0)}\|u\| \leq C'\langle\lambda\rangle^{-\frac{1}{2}}\|(K^L_\pm - i\lambda)u\|^2\,.$$

The case when $L_\pm$ is a geometric Kramers-Fokker-Planck operator on a general riemannian compact manifold will be studied in Section 6.
The general boundary conditions at $q = 0$, are formulated in terms of an operator $(A, D(A))$ defined in $L^2(\mathbb{R}, |p|dp; \mathfrak{L})$. They are written according to the introduction as

$$\gamma_{odd}u = \mathrm{sign}\,(p)A\gamma_{ev}u \quad , \quad \gamma u(p) = u(0,p)\,.$$

Remember the Definition 4.1 for the unitary involution j on $\mathfrak{L}$, for $\gamma_{ev} = \Pi_{ev}\gamma$ and $\gamma_{odd} = \Pi_{odd}\gamma$ for $\gamma \in L^2(\mathbb{R}, |p|dp; \mathfrak{L})$. All these operators j, Π_{ev} and Π_{odd} commute with $L_\pm$. We need some compatibility between $(A, D(A))$ and the involution j via $\Pi_{ev,odd}$, and we focus on the case $D(A) = L^2(\mathbb{R}, |p|dp; \mathfrak{L})$.

Hypothesis 4. *The operator A, $L^2(\mathbb{R}, |p|dp; \mathfrak{L})$, is a bounded accretive operator which commutes with the orthogonal projections Π_{ev} and $\Pi_{odd} = 1 - \Pi_{ev}$. The norm of A in $\mathcal{L}(L^2(\mathbb{R}, |p|dp; \mathfrak{L}))$ is simply written $\|A\|$.*
We assume additionally

either $c_A = \min \sigma(\mathbb{R}\mathrm{e}\; A) > 0$*;*

or $A = 0$.

Mixed boundary conditions with $A = A_1 \oplus 0$ when all the operators are block diagonal in $\mathfrak{L} = \mathfrak{L}_1 \oplus \mathfrak{L}_0$ will follow at once from the analysis of the distinct two cases. This will cover all the applications that we have in mind. More generally vanishing A's are not considered in this article.

5.2 Maximal accretivity

In this section we check the maximal accretivity of the realizations of

$$P_\pm^L = \pm p\partial_q + \frac{1}{2} + \mathcal{O} + L_\pm$$

on $\mathbb{R}^2_- = \mathbb{R}_- \times \mathbb{R}$, with boundary conditions given by $\gamma_{odd}u = \pm\,\mathrm{sign}\,(p)A\gamma_{ev}u$. The case $A = 0$ was treated in Proposition 4.8 without specifying $\gamma u \in L^2(\mathbb{R}, |p|dp; \mathfrak{L})$. This last point will be done in Subsection 5.3. We now focus on the case $A \neq 0$ of Hypothesis 4.
The strategy is a reduction to the boundary $\{q = 0\}$, of the equation

$$P_\pm^L u = f \quad \text{in } L^2(\mathbb{R}^2_-, dqdp; \mathfrak{L})\,, \quad \gamma_{odd}u = \pm\,\mathrm{sign}\,(p)A\gamma_{ev}u \quad \text{in } L^2(\mathbb{R}, |p|dp; \mathfrak{L})\,,$$

where the absence of a Calderon projector, will be compensated by the nice properties of $K^L_{\pm,1}$ (Hypothesis 1 for $L_\pm$ suffices here) combined with Hypothesis 4.

Proposition 5.1. *Assume Hypothesis 1 and Hypothesis 4 with* $A \neq 0$ *. Then the operator* $K^L_{\pm,A} - 1$ *defined by*

$$D(K^L_{\pm,A}) = \left\{ \begin{array}{ll} u \in L^2(\mathbb{R}_-, dq; \mathcal{H}^{1,L})\,, & P^L_{\pm} u \in L^2(\mathbb{R}^2_-, dqdp; \mathfrak{L}) \\ & \gamma_{ev} u \in L^2(\mathbb{R}, |p|dp; \mathfrak{L}) \\ & \gamma_{odd} u = \pm\, sign\,(p) A \gamma_{ev} u \end{array} \right\}\,,$$

$$\forall u \in D(K^L_{\pm,A})\,,\ K^L_{\pm,A} u = P^L_{\pm} u = (\pm p.\partial_q + \tfrac{1}{2} + \mathcal{O} + L)u\,,$$

is densely defined and maximal accretive in $L^2(\mathbb{R}^2_-, dqdp; \mathfrak{L})$ *.*
Any $u \in D(K^L_{\pm,A})$ *satisfies the integration by part identity*

$$\|u\|^2_{L^2(\mathbb{R}_-,dq;\mathcal{H}^{1,L})} + \mathrm{Re}\,\langle \gamma_{ev} u\,, A\gamma_{ev} u\rangle_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} = \mathrm{Re}\,\langle u\,,\, K^L_{\pm,A} u\rangle\,. \tag{55}$$

Moreover for any $\lambda \in \mathbb{R}$ *the operators* $(K^L_{\pm,A} - i\lambda)^{-1}$ *and* $\gamma\circ(K^L_{\pm,A} - i\lambda)^{-1}$ *have continuous extensions from* $L^2(\mathbb{R}_-, dq; \mathcal{H}^{-1,L})$ *respectively to* $L^2(\mathbb{R}_-, dq; \mathcal{H}^{1,L})$ *and* $L^2(\mathbb{R}, |p|dp; \mathfrak{L})$ *, with* λ*-dependent norms.*
Finally the adjoint of $K^L_{\pm,A}$ *is* $K^L_{\mp,A^*}$ *.*

Below is a more convenient (and more usual) rewriting of the boundary conditions.

Lemma 5.2. *Let* $(A, D(A))$ *be a maximal accretive operator in* $L^2(\mathbb{R}, |p|dp; \mathfrak{L})$ *. For* $\gamma \in L^2(\mathbb{R}, |p|dp; \mathfrak{L})$ *, the two relations*

$$\begin{array}{rl} & \Pi_{odd}\gamma = \pm\, sign\,(p) A \Pi_{ev}\gamma \quad (\Pi_{ev}\gamma \in D(A))\,, \\ \textit{and} & \Pi_{\mp}\gamma = \dfrac{(1-A)}{(1+A)} \Pi_{\pm}\gamma \end{array}$$

are equivalent.
When A fulfills Hypothesis 4,

$$\|\frac{1-A}{1+A}\|_{\mathcal{L}(L^2(\mathbb{R},|p|dp;\mathfrak{L}))} \leq \left(1 + \frac{2c_A}{1+\|A\|^2}\right)^{-\frac{1}{2}} < 1\,.$$

Proof. With

$$\Pi_+ = \Pi_{ev} + \mathrm{sign}\,(p)\Pi_{odd} \quad , \quad \Pi_- = \Pi_{ev} - \mathrm{sign}\,(p)\Pi_{odd}\,,$$

one gets

$$\begin{aligned} \left(\Pi_{\mp}\gamma = \frac{1-A}{1+A}\Pi_{\pm}\gamma\right) \quad &\Leftrightarrow \quad \left(\frac{2A}{1+A}\Pi_{ev}\gamma = \pm\frac{1}{1+A}\,\mathrm{sign}\,(p)\Pi_{odd}\gamma\right) \\ &\Leftrightarrow \quad (A\Pi_{ev}\gamma = \pm\,\mathrm{sign}\,(p)\Pi_{odd}\gamma \text{ in } D(A^*)') \\ &\overset{\Pi_{odd}\gamma \in L^2(\mathbb{R},|p|dp;\mathfrak{L})}{\Leftrightarrow} \quad \left(\begin{array}{l} \Pi_{ev}\gamma \in D(A) \\ \Pi_{odd}\gamma = \pm\,\mathrm{sign}\,(p) A \Pi_{ev}\gamma \end{array}\right)\,. \end{aligned}$$

When A is bounded and $\mathbb{R}\mathrm{e}\, A \geq c_A > 0$, the inequalities

$$\begin{aligned}
\|(1+A)u\|^2_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} - \|(1-A)u\|^2_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} &= 2\,\mathbb{R}\mathrm{e}\,\langle u\,,\, Au\rangle_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} \\
&\geq 2c_A^2\|u\|^2_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} \\
(1+\|A\|^2)\|u\|^2_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} &\geq \|(1-A)u\|^2_{L^2(\mathbb{R},|p|dp;\mathfrak{L})}\,,
\end{aligned}$$

yields

$$(1+\frac{2c_A}{1+\|A\|^2})\|(1-A)u\|^2_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} \leq \|(1+A)u\|^2_{L^2(\mathbb{R},|p|dp;\mathfrak{L})}\,.$$

□

Proof of Proposition 5.1. The domain contains $\mathcal{C}_0^\infty(\mathbb{R}^2_-;D(L))$ which is dense in $L^2(\mathbb{R}^2_-, dqdp;\mathfrak{L})$.
When u belongs to $D(K^L_{\pm,A})$, it satisfies all the assumptions of Proposition 4.5. We obtain

$$\begin{aligned}
\mathbb{R}\mathrm{e}\,\langle u\,,\, K^L_{\pm,A}u\rangle &\geq \|u\|^2_{L^2(\mathbb{R}_-,dq;\mathcal{H}^{1,L})} \pm \mathbb{R}\mathrm{e}\,\langle \gamma_{ev}u\,,\, \mathrm{sign}\,(p)\gamma_{odd}u\rangle_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} \\
&\geq \|u\|^2_{L^2(\mathbb{R}_-,dq;\mathcal{H}^{1,L})} + \mathbb{R}\mathrm{e}\,\langle \gamma_{ev}u\,,\, A\gamma_{ev}u\rangle_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} \\
&\geq \|u\|^2_{L^2(\mathbb{R}_-,dq;\mathcal{H}^{1,L})} \geq \|u\|^2\,.
\end{aligned}$$

and $K^L_{\pm,A}-1$ is accretive.
Let f belong to $L^2(\mathbb{R}^2_-, dqdp;\mathfrak{L})$, we want to find $u \in D(K^L_{\pm,A})$ such that $(K^L_{\pm,A} - i\lambda)u = f$. By Proposition 4.8, $u_1 = (K^L_{\pm,1} - i\lambda)^{-1}f$ belongs to $L^2(\mathbb{R}_-, dq;\mathcal{H}^{1,L})$ with $\gamma u_1 \in L^2(\mathbb{R},|p|dp;\mathfrak{L})$, $\gamma_{ev}u_1 = \pm\,\mathrm{sign}\,(p)\gamma_{odd}u_1$ or $\Pi_\mp\gamma u_1 = 0$.
Set $u = u_1 + v$ and the equation becomes

$$\begin{aligned}
&(P^L_\pm - i\lambda)v = 0 \quad , \quad v \in L^2(\mathbb{R}_-, dq;\mathcal{H}^{-1,L})\,, \\
&\Pi_\mp\gamma v = \Pi_\mp\gamma u = (\frac{1-A}{1+A})\Pi_\pm\gamma u = \frac{1-A}{1+A}\Pi_\pm\gamma u_1 + \frac{1-A}{1+A}\Pi_\pm\gamma v\,.
\end{aligned}$$

According to Definition 4.14, the problem is solved when $\Pi_\pm\gamma v$ is a solution to

$$\gamma = C^L_\pm(\lambda)(\frac{1-A}{1+A}\Pi_\pm\gamma u_1 + \frac{1-A}{1+A}\gamma) \quad , \quad \gamma \in \Pi_{ev}L^2(\mathbb{R},|p|dp;\mathfrak{L})\,.$$

Owing to $\|C_{\pm}^{L}(\lambda)\|\leq 1$ and the estimate $\|\frac{1-A}{1+A}\|\leq (1+\frac{2c_A}{1+\|A\|^2})^{-\frac{1}{2}}$ of Lemma 5.2, the above equation admits a unique solution $\gamma_{\pm}$, with

$$\|\gamma_{\pm}\|_{L^2(\mathbb{R},|p|dp;\mathfrak{L})}\leq C_A\|\Pi_{\pm}\gamma u_1\|_{L^2(\mathbb{R},|p|dp;\mathfrak{L})}\leq C'_A\|f\|_{L^2(\mathbb{R}_-,dq;\mathcal{H}^{-1,L})}\,.$$

Take

$$u=(K_{\pm,1}^{L}-i\lambda)^{-1}f+R_{\pm}^{L}(\lambda)\left[\frac{1-A}{1+A}\Pi_{\pm}\gamma u_1+\frac{1-A}{1+A}\gamma_{\pm}\right]\,.$$

It belongs to $L^2(\mathbb{R}_-,dq;\mathcal{H}^{1,L})$, solves $(P_{\pm}^{L}-i\lambda)u=f$. Its trace, as the sum of two terms, belongs to $L^2(\mathbb{R},|p|dp;\mathfrak{L})$ and solves

$$\Pi_{\mp}\gamma(u)=\frac{1-A}{1+A}\Pi_{\pm}\gamma u_1+\frac{1-A}{1+A}\gamma_{\pm}=\frac{1-A}{1+A}\Pi_{\pm}\gamma u\,.$$

This ends the proof of the maximal accretivity of $K_{\pm,A}^{L}$.
The function $u=(K_{\pm,A}-i\lambda)^{-1}f$ solves the boundary value problem (48) with $f_{\partial}=\Pi_{\mp}\gamma u$. Therefore Proposition 4.12 now provides the integration by part equality (55)

$$\begin{aligned}\mathbb{R}\mathrm{e}\,\langle u\,,\,(K_{\pm,A}^{L}-i\lambda)u\rangle=\frac{1}{4}\|\Pi_{+}\gamma u\|^2_{L^2(\mathbb{R},|p|dp;\mathfrak{L})}-\frac{1}{4}\|\Pi_{-}\gamma u\|^2_{L^2(\mathbb{R},|p|dp;\mathfrak{L})}\\ +\|u\|^2_{L^2(\mathbb{R}_-,dq;\mathcal{H}^{1,L})}\quad,\end{aligned}$$

after using (46).
This identity (55) combined with Hypothesis 4 ($A\neq 0$) implies

$$\begin{aligned}&\frac{c_A}{2}\|\gamma_{ev}u\|^2+\frac{c_A}{2\|A\|^2}\|\gamma_{odd}u\|^2+\|u\|^2_{L^2(\mathbb{R}_-,dq;\mathcal{H}^{1,L})}\\ &\qquad\leq\|u\|_{L^2(\mathbb{R}_-,dq;\mathcal{H}^1_L)}\|(K_{\pm,A}^{L}-i\lambda)u\|_{L^2(\mathbb{R}_-,dq;\mathcal{H}^{-1,L})}\,,\end{aligned}$$

which ensures the continuous extensions

$$\begin{aligned}&(K_{\pm,A}^{L}-i\lambda)^{-1}:L^2(\mathbb{R}_-,dq;\mathcal{H}^{-1,L})\to L^2(\mathbb{R},dq;\mathcal{H}^{1,L})\\ &\gamma(K_{\pm,A}^{L}-i\lambda)^{-1}:L^2(\mathbb{R}_-,dq;\mathcal{H}^{-1,L})\to L^2(\mathbb{R},|p|dp;\mathfrak{L})\,.\end{aligned}$$

Two functions $u\in D(K_{+,A}^{L})$ and $v\in(K_{-,A^*}^{L})$ fulfill the conditions of Proposition 4.3 (second case) and we get

$$\begin{aligned}\langle v\,,\,K^L_{+,A}u\rangle &= \langle v\,,\,P^L_+u\rangle\\ &= \langle \gamma v\,,\,\mathrm{sign}\,(p)\gamma u\rangle_{L^2(\mathbb{R},|p|dp;\mathfrak{L})}+\langle P^L_-v\,,\,u\rangle\\ &= \langle \gamma_{ev}v\,,\,\mathrm{sign}\,(p)\gamma_{odd}u\rangle_{L^2(\mathbb{R},|p|dp;\mathfrak{L})}\\ &\qquad+\langle \mathrm{sign}\,(p)\gamma_{odd}v\,,\,\gamma_{ev}u\rangle_{L^2(\mathbb{R},|p|dp;\mathfrak{L})}+\langle K^L_{-,A^*}v\,,\,u\rangle\\ &= \langle -A^*\gamma_{ev}v\,,\,\gamma_{ev}u\rangle_{L^2(\mathbb{R},|p|dp;\mathfrak{L})}\\ &\qquad+\langle \gamma_{ev}v\,,\,A\gamma_{ev}u\rangle_{L^2(\mathbb{R},|p|dp;\mathfrak{L})}+\langle K^L_{-,A^*}v\,,\,u\rangle\\ &= \langle K^L_{-,A^*}v\,,\,u\rangle\,.\end{aligned}$$

This proves $K^L_{-,A^*}\subset (K^L_{+,A})^*$ and therefore the equality, because K^L_{-,A^*} is maximal accretive. □

5.3 Half-space and whole space problem

Definition 5.3. *Let $\mathfrak{f}$ be one of the Hilbert spaces $\mathfrak{L}$, $D(L_\pm)$ or $D(L_\pm)'$, which are endowed with the involution j (see Definition 4.1). Let $\mathcal{H}^s$, $s\in\mathbb{R}$, be the space $\mathcal{H}^s=\left\{u\in S'(\mathbb{R})\,,\,(\frac{1}{2}+\mathcal{O})^{s/2}u\in L^2(\mathbb{R},dp)\right\}$.*
The operator $\Sigma: L^2(\mathbb{R}_-,dq;\mathcal{H}^s\otimes\mathfrak{f})\to L^2(\mathbb{R},dq;\mathcal{H}^s\otimes\mathfrak{f})$ is defined by

$$\Sigma u(q,p)=\begin{cases} u(q,p) & \text{if } q<0\,,\\ ju(-q,-p) & \text{if } q>0\,.\end{cases}$$

The operator $\tilde{\Sigma}: L^2(\mathbb{R},dq;\mathcal{H}^s\otimes\mathfrak{f})\to L^2(\mathbb{R},dq;\mathcal{H}^s\otimes\mathfrak{f})$ is defined by

$$\tilde{\Sigma}u(q,p)=ju(-q,-p)\,.$$

Proposition 5.4. *With Hypothesis 2, assume that $u\in L^2(\mathbb{R}_-,dq;\mathcal{H}^{1,L})\subset L^2(\mathbb{R}^2_-,dqdp;\mathfrak{L})$ solves $(P^L_\pm-i\lambda)u=f\in L^2(\mathbb{R}_-,dq;\mathcal{H}^{-1,L})\subset L^2(\mathbb{R}_-,dq;\mathcal{H}^{-1}\otimes D(L_\mp)')$ with $\lambda\in\mathbb{R}$. Then Σu solves*

$$(P^L_\pm-i\lambda)\Sigma u=\Sigma f\mp 2p(\gamma_{odd}u)\delta_0(q)\,,$$

in $\mathcal{S}'(\mathbb{R}^2;D(L_\mp)')$.
Moreover if $v\in L^2(\mathbb{R},dq;\mathcal{H}^{1,L})$ solves

$$(P^L_\pm-i\lambda)v=\Sigma f\mp 2p(\gamma_{odd}u)\delta_0(q)$$

in $\mathcal{S}'(\mathbb{R}^2;D(L_\mp)')$, then $v=\Sigma u$.

Proof. When $u \in L^2(\mathbb{R}_-, dq; \mathcal{H}^{1,L}) \subset L^2(\mathbb{R}^2_-, dqdp; \mathfrak{L})$ solves $(P^L_\pm - i\lambda)u = f$ with $f \in L^2(\mathbb{R}_-, dq; \mathcal{H}^{-1,L}) \subset L^2(\mathbb{R}^2_-, dqdp; \mathfrak{L})$, it belongs to $\mathcal{E}^{L,\pm}_{\mathbb{R}_-} \subset \mathcal{E}_{\mathbb{R}_-}(D(L_\mp)')$ and solves

$$(\pm p\partial_q + \frac{1}{2} + \mathcal{O} - i\lambda)u = f - L_\pm u \quad \text{in } L^2_\pm(\mathbb{R}^2_-, dqdp; D(L_\mp)'),$$

with $\gamma u \in L^2(\mathbb{R}, |p|dp; D(L_\mp)')$.
With $[(1+L_\pm)^{-1}, j] = 0$ and

$$[\pm p\partial_q + \frac{1}{2} + \mathcal{O}][ju(-q,-p)] = j[(\pm p\partial_q + \frac{1}{2} + \mathcal{O})u](-q,-p) \quad \text{in } \mathbb{R}^2 \setminus \{q = 0\},$$

we obtain

$$\begin{aligned}(\pm p\partial_q + \frac{1}{2} + \mathcal{O})(\Sigma u) &= \Sigma f - L_\pm \Sigma u \pm p\left[\Sigma u(0^+,p) - \Sigma u(0^-,p)\right]\delta_0(q)\\ &= \Sigma f - L_\pm \Sigma u \mp 2p\gamma_{odd} u \delta_0(q),\end{aligned}$$

in $\mathcal{S}'(\mathbb{R}^2; D(L^*)')$, which is what we seek.
When $v \in L^2(\mathbb{R}, dq; \mathcal{H}^{1,L})$ is another solution to $(P^L_\pm - i\lambda)v = \Sigma f \mp 2p\gamma_{odd} u\delta_0(q)$, then the difference $w = v - \Sigma u$ belongs to $L^2(\mathbb{R}, dq; \mathcal{H}^{1,L})$ and solves $(P^L_\pm - i\lambda)w = 0$. Therefore it belongs to $D(K^L_\pm)$ and solves $K^L_\pm w = 0$, which implies $w = 0$. $\square$

From the previous result, one infers the equivalence

$$(u = (K^L_{\pm,0} - i\lambda)^{-1} f) \Leftrightarrow (\Sigma u = (K^L_\pm - i\lambda)^{-1}\Sigma f) \tag{56}$$

which combined with Hypothesis 2 provides good estimates for $(K^L_{\pm,0} - i\lambda)^{-1}$ with respect to $\lambda \in \mathbb{R}$. Actually this proves $\gamma u \in L^2(\mathbb{R}, |p|dp; \mathfrak{L})$ when $u \in D(K^L_{\pm,0})$ and it solves completely the case $A = 0$.

Proposition 5.5. *With Hypothesis 2, there exists a constant $C > 0$ such that*

$$\begin{aligned}\langle\lambda\rangle^{\frac{1}{2}}\|u\| + \|(\frac{1}{2} + \mathcal{O} + \mathrm{Re}\ L)u\| + \|(\pm p\partial_q + \frac{L_\pm - L_\mp}{2} - i\lambda)u\| + \|\Sigma u\|_{\mathcal{Q}_0}\\ + \langle\lambda\rangle^{\frac{1}{4}}\|u\|_{L^2(\mathbb{R}_-, dq; \mathcal{H}^{1,L})} + \langle\lambda\rangle^{\frac{1}{4}}\|\gamma u\|_{L^2(\mathbb{R}, |p|dp; \mathfrak{L})} \leq C\|(K^L_{\pm,0} - i\lambda)u\|\end{aligned}$$

holds for all $\lambda \in \mathbb{R}$ and all $u \in D(K^L_{\pm,0})$.
If $\mathcal{D}$ is dense in $D(K^L_\pm) \subset L^2(\mathbb{R}, dqdp; \mathfrak{L})$ endowed with the graph norm, with $\tilde{\Sigma}\mathcal{D} = \mathcal{D}$, then the set $\left\{u \in L^2(\mathbb{R}^2_-, dqdp; \mathfrak{L}), \Sigma u \in \mathcal{D}\right\}$ is dense in $D(K^L_{\pm,0})$ endowed with its graph norm.

Proof. With $2\|(K^L_{\pm,0}-i\lambda)u\|^2=\|(K_\pm-i\lambda)\Sigma u\|^2$, the upper bound of

$$\sqrt{2}\left[\langle\lambda\rangle\|u\|+\|(\frac{1}{2}+\mathcal{O}+\mathbb{R}\mathrm{e}\ L)u\|+\|(\pm p\partial_q+\frac{L_\pm-L_\mp}{2}-i\lambda)u\|\right]$$
$$=\langle\lambda\rangle\|\Sigma u\|+\|(\frac{1}{2}+\mathcal{O}+\mathbb{R}\mathrm{e}\ L)\Sigma u\|+\|(\pm p\partial_q+\frac{L_\pm-L_\mp}{2}-i\lambda)\Sigma u\|$$

and $\|\Sigma u\|_{\mathcal{Q}_0}$ are assumed in Hypothesis 2.
The upper bound for $\langle\lambda\rangle^{\frac{1}{4}}\|u\|_{L^2(\mathbb{R}_-,dq;\mathcal{H}^{1,L})}$ comes from

$$\|v\|^2_{L^2(\mathbb{R}_-,dq;\mathcal{H}^{1,L})}\leq\|v\|\|(\frac{1}{2}+\mathcal{O}+\mathbb{R}\mathrm{e}\ L)v\|\,.$$

The upper bound for $\langle\lambda\rangle^{\frac{1}{4}}\|\gamma u\|_{L^2(\mathbb{R},|p|dp;\mathfrak{L})}$ is proved in Proposition 5.6 below.
For the density of $\left\{u\in L^2(\mathbb{R}^2_-,dqdp;\mathfrak{L})\,,\Sigma u\in\mathcal{D}\right\}$, we start from the relations

$$\tilde{\Sigma}\Sigma=\Sigma\quad,\quad K^L_\pm\Sigma=\Sigma K^L_{\pm,0}\,.$$

When $u\in D(K^L_{\pm,0})$, the symmetrized function Σu belongs to $D(K^L_\pm)$ and can be approximated by a sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathcal{D}$.
With $\tilde{\Sigma}\mathcal{D}=\mathcal{D}$, we deduce

$$0=\lim_{n\to\infty}\|\frac{1+\tilde{\Sigma}}{2}u_n-\frac{1+\tilde{\Sigma}}{2}\Sigma u\|=\lim_{n\to\infty}\|\frac{1+\tilde{\Sigma}}{2}u_n-\Sigma u\|\,,$$
$$0=\lim_{n\to\infty}\|\frac{1+\tilde{\Sigma}}{2}K^L_\pm u_n-\frac{1+\tilde{\Sigma}}{2}\Sigma K^L_{\pm,0}u\|=\lim_{n\to\infty}\|\frac{1+\tilde{\Sigma}}{2}K^L_\pm u_n-\Sigma K^L_{\pm,0}u\|\,.$$

The function $v_n=1_{\mathbb{R}_-}(q)\frac{1+\tilde{\Sigma}}{2}u_n$ satisfies

$$\Sigma v_n=\frac{1+\tilde{\Sigma}}{2}u_n\in\mathcal{D}$$

and the commutation $P^L_\pm\tilde{\Sigma}=\tilde{\Sigma}P^L_\pm$ implies

$$K^L_\pm\Sigma v_n=K^L_\pm\left[\frac{1+\tilde{\Sigma}}{2}u_n\right]=\frac{1+\tilde{\Sigma}}{2}K^L_\pm u_n\,.$$

We deduce $v_n\in D(K^L_{\pm,0})$, $\Sigma K^L_{\pm,0}v_n=\frac{1+\tilde{\Sigma}}{2}K^L_\pm u_n$ and

$$\lim_{n\to\infty}\|v_n-u\|+\|K^L_{\pm,0}(v_n-u)\|=\frac{1}{2}\lim_{n\to\infty}\|\Sigma v_n-\Sigma u\|+\|\Sigma K^L_{\pm,0}(v_n-u)\|=0\,,$$

while we have checked $\Sigma v_n\in\mathcal{D}$ for all $n\in\mathbb{N}$. □

With (56) the estimate of $\langle\lambda\rangle^{\frac{1}{4}}\|\gamma u\|_{L^2(\mathbb{R},|p|dp;\mathfrak{L})}$ is a variant of Proposition A.4 and the proof follows the same lines.

Proposition 5.6. *Assume Hypothesis 2. Any $u\in D(K_\pm^L)$ has a trace $\gamma u = u(0,p)\in L^2(\mathbb{R},|p|dp;\mathfrak{L})$ with the estimate*

$$\forall\lambda\in\mathbb{R}\,,\quad \langle\lambda\rangle^{\frac{1}{2}}\|u(q=0,p)\|^2_{L^2(\mathbb{R},|p|dp;\mathfrak{L})}\leq C'\|(K_\pm^L-i\lambda)u\|^2\,.$$

When $u\in L^2(\mathbb{R}^2,dqdp;\mathfrak{L})$ solves $(P_\pm^L-i\lambda)u=\gamma(p)\delta_0(q)$ in $\mathcal{S}'(\mathbb{R}^2;D(L^*)')$ with $\gamma\in L^2(\mathbb{R},\frac{dp}{|p|};\mathfrak{L})$ and $\lambda\in\mathbb{R}$, then

$$\langle\lambda\rangle^{\frac{1}{2}}\|u\|^2\leq C''\|\gamma\|^2_{L^2(\mathbb{R},\frac{dp}{|p|};\mathfrak{L})}\,.$$

Proof. We first take $u\in D(K_\pm)\otimes^{alg}D(L_\pm)$ and $\lambda\in\mathbb{R}$. We compute

$$\begin{aligned}
\int_{\mathbb{R}}|p||u(0,p)|^2_{\mathfrak{L}}\,dp &= \pm2\,\mathrm{Re}\int_{-\infty}^{0}\langle(\pm p\partial_q+i\mathrm{Im}\,L-i\lambda)u\,,\mathrm{sign}\,(p)u\rangle_{\mathfrak{L}}\,dp\\
&= \pm2\,\mathrm{Re}\,\langle(i\mathrm{Im}\,K_\pm^L-i\lambda)u\,,\,1_{\mathbb{R}_-}(q)\,\mathrm{sign}\,(p)u\rangle\\
&\leq 2\|(i\mathrm{Im}\,K_\pm^L-i\lambda)u\|\|u\|\leq C\langle\lambda\rangle^{-\frac{1}{2}}\|(K_\pm^L-i\lambda)u\|^2\,.
\end{aligned}$$

The results extends to any $u\in D(K_\pm^L)$ by density.
A similar result is valid for $(K_\pm^L)^*=K_\mp^L$, while $\mathcal{S}(\mathbb{R}^2;D(L_\mp))$ is a core for $K_\mp^L$. Thus assuming

$$\forall v\in\mathcal{S}(\mathbb{R}^2;D(L_\mp))\,,\langle u\,,\,(K_\mp^L+i\lambda)v\rangle=\int_{\mathbb{R}}\langle\gamma(p)\,,\,v(0,p)\rangle_{\mathfrak{L}}\,dp\,,$$

leads to

$$\begin{aligned}
\left|\langle u\,,\,(K_\mp^L+i\lambda)v\rangle\right| &\leq \||p|^{-\frac{1}{2}}\gamma\|_{L^2(\mathbb{R},dp;\mathfrak{L})}\||p|^{\frac{1}{2}}\gamma v\|_{L^2(\mathbb{R},dp;\mathfrak{L})}\\
&\leq \|\gamma\|_{L^2(\mathbb{R},\frac{dp}{|p|};\mathfrak{L})}\|\gamma v\|_{L^2(\mathbb{R},|p|dp;\mathfrak{L})}\\
&\leq C\langle\lambda\rangle^{-\frac{1}{4}}\|\gamma\|_{L^2(\mathbb{R},\frac{dp}{|p|};\mathfrak{L})}\|(K_\mp^L+i\lambda)v\|\,,
\end{aligned}$$

for all $v\in\mathcal{S}(\mathbb{R}^2;D(L_\mp))$. Since $\mathcal{S}(\mathbb{R}^2;D(L_\mp))$ is dense in $D(K_\mp^L)$ and $(K_\mp^L+i\lambda)$ is an isomorphism from $D(K_\mp^L)$ to $L^2(\mathbb{R}^2,dqdp;\mathfrak{L})$, we deduce

$$\forall f\in L^2(\mathbb{R}^2,dqdp;\mathfrak{L})\,,\quad|\langle u\,,\,f\rangle|\leq C\langle\lambda\rangle^{-\frac{1}{4}}\|\gamma\|_{L^2(\mathbb{R},\frac{dp}{|p|};\mathfrak{L})}\|f\|\,,$$

which proves the estimate $\langle\lambda\rangle^{\frac{1}{2}}\|u\|^2\leq C''\|\gamma\|^2_{L^2(\mathbb{R},\frac{dp}{|p|};\mathcal{L})}$. $\square$

Remark 5.7. *For any $\lambda \in \mathbb{R}$, the only possible solution to*

$$(P_{\pm} - i\lambda)u = 0 \quad , \quad \gamma u \in \Pi_{odd} L^2(\mathbb{R}, |p|dp; \mathfrak{L})$$

is the trivial solution $u = 0$. Actually, if $u \in L^2(\mathbb{R}_-, dq; \mathcal{H}^{1,L})$ is a solution, it has a trace $\gamma \in L^2(\mathbb{R}, |p|dp; D(L_{\mp})')$ and the function $\tilde{u}(q,p) = u(q,p)1_{\mathbb{R}_-}(q)$ solves

$$(P_{\pm} - i\lambda)\tilde{u} = -p\gamma\delta_0(q) \quad \text{in } \mathcal{S}'(\mathbb{R}^2; D(L_{\mp})') \,.$$

By Proposition 5.4, we deduce $\Sigma u = 2\tilde{u}$ and $\tilde{u}\big|_{\{q>0\}} \equiv 0$ implies $\Sigma u = 0$ and $u = 0$.

5.4 Resolvent estimates

5.4.1 Trace estimates

Proposition 5.8. *Assume Hypotheses 2 and 4 with $A \neq 0$. There exists a constant $C_{1,L,A} > 0$ such that the estimate*

$$\forall f \in L^2(\mathbb{R}^2_-, dqdp; \mathfrak{L}) \,, \quad \|\gamma(K^L_{\pm,A} - i\lambda)^{-1} f\|_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} \leq C_{1,L,A}\langle\lambda\rangle^{-\frac{1}{4}}\|f\| \,,$$

holds uniformly w.r.t $\lambda \in \mathbb{R}$.

Proof. For $f \in L^2(\mathbb{R}^2_-, dqdp; \mathfrak{L})$, the function $u = (K^L_{\pm,A} - i\lambda)^{-1} f \in D(K^L_{\pm,A})$ solves

$$\begin{aligned} & (P^L_{\pm} - i\lambda)u = f \quad , \quad \gamma_{odd} u = f_{\partial} \\ \text{with} \quad & f_{\partial} = \gamma_{odd} u = \pm \operatorname{sign}(p) A \gamma_{ev} u \quad , \quad \gamma u \in L^2(\mathbb{R}, |p|dp; \mathfrak{L}) \,. \end{aligned}$$

The difference $v = u - (K_{\pm,0} - i\lambda)^{-1} f$ satisfies $v \in L^2(\mathbb{R}_-, dq; \mathcal{H}^{1,L})$, $\gamma v \in L^2(\mathbb{R}, |p|dp; \mathfrak{L})$ and solves

$$(P_{\pm} - i\lambda)v = 0 \quad , \quad \gamma_{odd} v = f_{\partial} \,.$$

Proposition 4.13 then says

$$\pm \operatorname{Re} \langle \gamma_{ev} v \,, \operatorname{sign}(p)\gamma_{odd} v\rangle_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} = \operatorname{Re} \langle \gamma_{ev} v \,, \pm \operatorname{sign}(p) f_{\partial}\rangle_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} \leq 0 \,.$$

The even part of the trace at $q = 0$ of $u = (K^L_{\pm,0} - i\lambda)^{-1} f + v$ equals

$$\gamma_{ev} u = \gamma_{ev}(K^L_{\pm,0} - i\lambda)^{-1} f + \gamma_{ev} v \,,$$

Taking the $L^2(\mathbb{R}, |p|dp; \mathfrak{L})$-scalar product with $\pm\,\mathrm{sign}\,(p) f_\partial = A\gamma_{ev} u$ says that the quantity

$$\mathbb{R}\mathrm{e}\, \langle A\gamma_{ev} u\,,\, \gamma_{ev} u\rangle_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} = \mathbb{R}\mathrm{e}\, \langle \pm\,\mathrm{sign}\,(p) f_\partial\,,\, \gamma_{ev} u\rangle_{L^2(\mathbb{R},|p|dp;\mathfrak{L})}$$

cannot be larger than

$$\begin{aligned}&\mathbb{R}\mathrm{e}\, \langle \pm\,\mathrm{sign}\,(p) f_\partial\,,\, \gamma_{ev}(K^L_{\pm,0} - i\lambda)^{-1} f\rangle_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} \\ &\qquad\qquad \leq \|\gamma_{odd} u\|_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} \|\gamma (K^L_{\pm,0} - i\lambda)^{-1} f\|_{L^2(\mathbb{R},|p|dp;\mathfrak{L})}\,.\end{aligned}$$

With Hypothesis 4 and $A \neq 0$, we deduce

$$\begin{aligned}\frac{c_A}{\|A\|^2}\|\gamma_{odd} u\|^2_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} &= \frac{c_A}{\|A\|^2}\|A\gamma_{ev} u\|^2_{L^2(\mathbb{R},|p|dp;\otimes\mathfrak{L})} \\ &\leq \mathbb{R}\mathrm{e}\, \langle A\gamma_{ev} u\,,\, \gamma_{ev} u\rangle_{L^2(\mathbb{R},|p|dp;\otimes\mathfrak{L})} \\ &\leq \|\gamma_{odd} u\|_{L^2(\mathbb{R},|p|dp;\otimes\mathfrak{L})} \|\gamma (K^L_{\pm,0} - i\lambda)^{-1} f\|_{L^2(\mathbb{R},|p|dp;\mathfrak{L})}\,.\end{aligned}$$

We infer from this

$$\begin{aligned}&\|\gamma_{odd} u\|_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} \leq \frac{\|A\|^2}{c_A} \|\gamma (K^L_{\pm,0} - i\lambda)^{-1} f\|_{L^2(\mathbb{R},|p|dp;\mathfrak{L})}\,, \\ \text{and}\qquad &\|\gamma_{ev} u\|_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} \leq \frac{\|A\|}{c_A} \|\gamma (K^L_{\pm,0} - i\lambda)^{-1} f\|_{L^2(\mathbb{R},|p|dp;\mathfrak{L})}\,,\end{aligned}$$

and we conclude by referring to Proposition 5.5. □

5.4.2 L^2-estimates

Proposition 5.9. *Assume Hypotheses 2 and 4 with $A \neq 0$. There exists a constant $C_{2,L,A} > 0$ such that*

$$\forall f \in L^2(\mathbb{R}^2_-, dqdp; \mathfrak{L})\,, \quad \|(K^L_{\pm,A} - i\lambda)^{-1} f\| \leq C_{2,L,A} \langle\lambda\rangle^{-\frac{1}{2}} \|f\|\,,$$

holds uniformly w.r.t $\lambda \in \mathbb{R}$.

Proof. Consider $u = (K^L_{\pm,A} - i\lambda)^{-1}$ with $f \in L^2(\mathbb{R}^2_-, dqdp; \mathfrak{L})$ and decompose it into

$$u = (K^L_{\pm,0} - i\lambda)^{-1} f + v\,.$$

The first term satisfies

$$\Sigma (K^L_{\pm,0} - i\lambda)^{-1} f = (K^L_{\pm} - i\lambda)(\Sigma f)$$

according to (56). We deduce

$$\|(K^L_{\pm,0}-i\lambda)^{-1}f\| \leq \frac{1}{\sqrt{2}}\|(K^L_{\pm}-i\lambda)^{-1}(\Sigma f)\| \leq \frac{C}{\sqrt{2}}\langle\lambda\rangle^{-\frac{1}{2}}\|f\|\,.$$

The second term v belongs to $L^2(\mathbb{R}_-, dq; \mathcal{H}^1)$ with $\gamma v \in L^2(\mathbb{R}, |p|dp; \mathfrak{L})$ and solves

$$(P_{\pm}-i\lambda)v = 0 \quad , \quad \gamma_{odd}v = \gamma_{odd}u\,.$$

Proposition 5.4 says that Σv solves

$$(P_{\mp}-i\lambda)\Sigma v = \mp 2p(\gamma_{odd}u)\delta_0(q)$$

and Proposition 5.6 implies

$$\begin{aligned}\|v\|^2 &\leq \frac{1}{2}\|\Sigma v\|^2 \leq \frac{C''}{2}\langle\lambda\rangle^{-\frac{1}{2}}\|p\gamma_{odd}u\|^2_{L^2(\mathbb{R},\frac{dp}{|p|};\mathfrak{L})} \\ &\leq \frac{C''}{2}\langle\lambda\rangle^{-\frac{1}{2}}\|\gamma u\|^2_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} \leq \frac{C''C'}{2}\langle\lambda\rangle^{-1}\|f\|^2\,.\end{aligned}$$

□

5.4.3 Regularity estimates

Proposition 5.10. *Assume Hypotheses 2 and 4 with $A \neq 0$. There exists a constant $C_{3,A,L} > 0$ such that*

$$\langle\lambda\rangle\|u\|^2+\langle\lambda\rangle^{\frac{1}{2}}\|u\|^2_{L^2(\mathbb{R}_-,dq;\mathcal{H}^{1,L})}+\langle\lambda\rangle^{\frac{1}{2}}\|\gamma u\|^2_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} \leq C_{3,A,L}\|(K^L_{\pm,A}-i\lambda)u\|^2\,,$$

holds for all $u \in D(K^L_{\pm,A})$ and all $\lambda \in \mathbb{R}$.
If $\Phi \in \mathcal{C}^\infty_b((-\infty,0])$ satisfies $\Phi(0)=0$, there exists $C_{L,\Phi} > 0$ such that

$$\|\Phi(q)[\frac{1}{2}+\mathcal{O}+\mathbb{R}\mathrm{e}\;L]u\| \leq C\|\Phi\|_{L^\infty}\|(K^L_{\pm,A}-i\lambda)u\| + C_{L,\Phi}\|u\|\,,$$

holds true when $\lambda \in \mathbb{R}$, $u \in D(K^L_{\pm,A})$ and C is the constant of Hypothesis 2.

Proof. **a)** The bound for $\langle\lambda\rangle\|u\|^2$ and $\langle\lambda\rangle^{1/2}\|\gamma_{odd}u\|^2_{L^2(\mathbb{R},|p|dp;\mathfrak{L})}$ are given in Proposition 5.9 and Proposition 5.8.

b) For $u \in D(K_A^L)$, the integration by part indentity of Proposition 5.1, combined with Hypothesis 4 ($A \neq 0$), gives

$$\begin{array}{rcl}\|u\|\|(K^L_{\pm,A} - i\lambda)u\| & \geq & \mathbb{R}\mathrm{e}\,\langle u\,,\,K^L_{\pm,A}u\rangle \\ & \geq & \|u\|^2_{L^2(\mathbb{R}_-,dq;\mathcal{H}^{1,L})} + \mathbb{R}\mathrm{e}\,\langle \gamma_{ev}u\,,\,A\gamma_{ev}u\rangle^2_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} \\ & \geq & \|u\|^2_{L^2(\mathbb{R}_-,dq;\mathcal{H}^{1,L})} + \dfrac{c_A}{2}\|\gamma_{ev}u\|^2_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} \\ & & \qquad\qquad + \dfrac{c_A}{2\|A\|^2}\|\gamma_{odd}u\|^2_{L^2(\mathbb{R},|p|dp;\mathfrak{L})}\,.\end{array}$$

With **a)** or Proposition 5.9, we deduce

$$\begin{array}{rcl}\|(K^L_{\pm,A} - i\lambda)u\|^2 & \geq & C^{-1}_{2,A,L}\langle\lambda\rangle^{1/2}\|u\|\|(K^L_{\pm,A} - i\lambda)uu\| \\ & \geq & C^{-1}_{A,L}\langle\lambda\rangle^{1/2}\left[\|u\|^2_{L^2(\mathbb{R}_-,dq;\mathcal{H}^{1,L})} + \|\gamma u\|^2_{L^2(\mathbb{R},|p|dp;\mathfrak{L})}\right]\,.\end{array}$$

c) For $\Phi \in \mathcal{C}^\infty_b(\overline{\mathbb{R}_-})$ such that $\Phi(0) = 0$, $\lambda \in \mathbb{R}$ and $u \in D(K^L_{\pm,A})$ with $(K^L_{\pm,A} - i\lambda)u = f$, write:

$$(P^L_\pm - i\lambda)(\Phi(q)u) = \Phi(q)(P^L_\pm - i\lambda)u \pm p\Phi'(q)u = \Phi(q)f \pm \Phi'(q)(pu)\,.$$

The assumption $\Phi(0) = 0$ implies

$$\begin{array}{ll} & (\Phi(q)u)1_{\mathbb{R}_-}(q) \in D(K^L_\pm) \\ \text{and} & (K^L_\pm - i\lambda)(\Phi(q)u1_{\mathbb{R}_-}(q)) = 1_{\mathbb{R}_-}(q)\left[\Phi(q)f \pm \Phi'(q)(pu)\right]\,.\end{array}$$

Hypothesis 2 implies

$$\begin{array}{rcl}\|(\dfrac{1}{2} + \mathcal{O} + \mathbb{R}\mathrm{e}\;L)\Phi(q)u\| & = & \|\dfrac{K^L_\pm + K^{L^*}_\mp}{2}(\Phi(q)u1_{\mathbb{R}_-}(q))\| \\ & \leq & C\|1_{\mathbb{R}_-}(q)\left[\Phi(q)f + \Phi'(q)(pu)\right]\| \\ & \leq & CM^0_\Phi\|(K^L_{\pm,A} - i\lambda)u\| + M^1_\Phi\|pu\| \\ & \leq & CM^0_\Phi\|(K^L_{\pm,A} - i\lambda)u\| + 2M^1_\Phi\|u\|_{L^2(\mathbb{R}_-,dq;\mathcal{H}^{1,L})}\,,\end{array}$$

where we have used $M^k_\Phi = \|\partial^k_q\Phi\|_{L^\infty}$. Again from

$$\|u\|\|(K^L_{\pm,A} - i\lambda)u\| \geq \mathbb{R}\mathrm{e}\,\langle u\,,\,K^L_{\pm,A}u\rangle \geq \|u\|^2_{L^2(\mathbb{R}_-,dq;\mathcal{H}^{1,L})}\,,$$

we deduce

$$\|(\frac{1}{2} + \mathcal{O} + \mathbb{R}\mathrm{e}\;L)\Phi(q)u\| \leq 2CM^0_\Phi\|(K^L_{\pm,A} - i\lambda)u\| + C_{L,\Phi}\|u\|\,,$$

by choosing simply $C_{L,\Phi} = \frac{2(M^1_\Phi)^2}{CM^0_\Phi}$. $\square$

Proposition 5.11. *Assume Hypotheses 2, 4 with $A \neq 0$ and Hypothesis 3. Then for any $u \in D(K^L_{\pm,A})$, Σu belongs to $\mathcal{Q}$ with the estimate*

$$\forall \lambda \in \mathbb{R}\,, \forall u \in D(K^L_{\pm,A})\,, \quad \langle\lambda\rangle^{\frac{1}{4}}\|\Sigma u\|_{\mathcal{Q}} \leq C_{\mathcal{Q},A,L}\|(K^L_{\pm,A} - i\lambda)u\|$$

Proof. Again we write that $\Sigma u = \Sigma(K^L_{\pm,A} - i\lambda)^{-1}f$ satisfies

$$(P^L_{\pm} - i\lambda)u = \Sigma f \mp 2p(\gamma_{odd}u)\delta_0(q)\,,$$

with $2p(\gamma_{odd}u) \in L^2(\mathbb{R}, \frac{dp}{|p|}; \mathfrak{L})$. According to Proposition 5.6, we call $v \in L^2(\mathbb{R}, dq; \mathcal{H}^{1,L})$ the solution to (52) with $\gamma = \mp 2p(\gamma_{odd}u)$ so that

$$\Sigma u = (K^L_{\pm} - i\lambda)^{-1}(\Sigma f) + v\,.$$

The first assumed inequality (53) in Hypothesis 3 gives

$$\|v\|_{\mathcal{Q}} \leq C\|\gamma_{odd}u\|_{L^2(\mathbb{R},|p|dp;\mathfrak{L})}$$

while Proposition 5.8 implies

$$\|\gamma_{odd}u\|_{L^2(\mathbb{R},|p|dp;\mathfrak{L})} \leq C_{1,L,A}\langle\lambda\rangle^{-\frac{1}{4}}\|f\|\,.$$

The second assumed inequality (54) in Hypothesis 3,

$$\|(K^L_{\pm} - i\lambda)^{-1}(\Sigma f)\|_{\mathcal{Q}} \leq C\langle\lambda\rangle^{-1/4}\|f\|\,,$$

yields the result. ☐

6 Geometric Kramers-Fokker-Planck operator

In [Leb1]citeLeb2, G. Lebeau developed the resolvent analysis for geometric (Kramers)-Fokker-Planck operators, which is the scalar model for the hypoelliptic Laplacian introduced by J.M. Bismut (see [BiLe][Bis05][Bis1]) and which will be considered in Section 9.2. Lebeau in [Leb1][Leb2] studied the case of compact manifolds but a partition of unity allows to extend it to the case of cylinders $Q = \mathbb{R} \times Q'$ when Q' is a compact manifold. We will check that Hypothesis 2 and Hypothesis 3 are satisfied by the geometric Kramers-Fokker-Planck operator on T^*Q when $Q = \mathbb{R} \times Q'$ is endowed with a specific

splitted metric. This will prepare the analysis of half-cylinders models for the boundary case. We also review some partition of unity arguments which will be used more extensively in Section 7 and extract the usefull information from the results of [Leb2].
For this presentation we stick to the case of the scalar geometric Kramers-Fokker-Planck equations. Tensorizing with a Hilbert space $\mathfrak{f}$ makes no difficulty in the end.

6.1 Notations and the geometric KFP-operator

We consider a d-dimensional Riemannian manifold (Q,g) (possibly with a boundary ∂Q) endowed with the metric $g(q)=(g_{ij}(q))_{1\leq i,j\leq d}$,

$$g(q;T)=\sum_{i,j=1}^{d} g_{ij}(q)T^iT^j = g_{ij}(q)T^iT^j = T^Tg(q)T\,,\quad T\in T_qQ\,,$$

with Einstein summation convention and the matricial writing for the last right-hand side. The q coordinates are written $(q^1\,\ldots,q^d)\in\mathbb{R}^d$ and when we use the matricial notation the vectors are column-vectors. The inverse tensor $g^{-1}(q)$ is denoted by $g^{-1}(q)=(g^{ij}(q))_{1\leq i,j\leq d}$ with

$$g^{ik}g_{kj}=\delta^i_j\,.$$

With the metric g are associated the Christoffel symbols

$$\Gamma^\ell_{ij}=\frac{1}{2}g^{\ell k}\left\{\partial_{q^i}g_{kj}+\partial_{q^j}g_{ik}-\partial_{q^k}g_{ij}\right\}=\Gamma^\ell_{ji}\,,$$

the Levi-Civita connection

$$\nabla_{\partial_{q^i}}\left(\partial_{q^j}\right)=\Gamma^\ell_{ij}\partial_{q^\ell}=(\Gamma_i)^\ell_j\partial_{q^j}\,, \tag{57}$$

with its adjoint version

$$\nabla_{\partial_{q^i}}\left(dq^j\right)=-\Gamma^j_{i\ell}dq^\ell\,, \tag{58}$$

and the Riemann curvature tensor, which is an $\mathrm{End}(TQ)$-valued two form,

$$\begin{aligned}&R=R_{jk}dq^j\wedge dq^k\\&R_{jk}=\frac{\partial\Gamma_j}{\partial q^k}-\frac{\partial\Gamma_k}{\partial q^j}+[\Gamma_j\,,\,\Gamma_k]\,.\end{aligned}$$

The geodesic flow in the tangent space TQ is obtained after minimizing

$$\frac{1}{2}\int_0^1 g(q;\dot\sigma)\ dt$$

with respect to $\sigma \in \mathcal{C}^2([0,1];Q)$ and is locally equivalent to

$$\ddot\sigma^\ell + \Gamma^\ell_{ij}\dot\sigma^i\dot\sigma^j = 0\,.$$

This is the Lagrangian version of the geodesic flow which has the Hamiltonian counterpart (see [AbMa] for example). The cotangent bundle over Q is denoted by $X = T^*Q$. The length of a covector $p_i dq^i \in T^*_q Q$ is given by $|p|_q^2 = g^{ij}(q)p_ip_j = p^T g^{-1}(q)p$. The Hamiltonian version of the geodesic flow is given by the Hamiltonian vector field $\mathcal{Y}_\mathcal{E}$ associated with the energy $\mathcal{E}(q,p) = \frac{1}{2}|p|_q^2$. With the symplectic form $dp_i \wedge dq^i$, it is given by

$$\mathcal{Y}_\mathcal{E} = p^T g^{-1}(q)\partial_q - \frac{1}{2}\partial_q\left[p^T g^{-1}(q)p\right]^T\partial_p = g^{ij}(q)p_i\partial_{q^j} - \frac{1}{2}[\partial_{q^k}g^{ij}(q)]p_ip_j\partial_{p_k}\,.$$

Owing to the relation

$$-\frac{1}{2}\partial_{q^k}g^{ij}p_ip_j = g^{i\ell}\Gamma^j_{k\ell}p_ip_j$$

it also equals

$$\mathcal{Y}_\mathcal{E} = g^{ij}(q)p_ie_j = p^T g^{-1}(q)e\,,$$

$$\text{with}\qquad e_j = \partial_{q^j} + \Gamma^\ell_{i,j}p_\ell\partial_{p_i}\quad,\quad e = \begin{pmatrix} e_1\\ \vdots \\ e_d\end{pmatrix}\,.$$

When $q \to \tilde q = \Phi(q)$ is a change of coordinate in a neighborhood of $q_0 \in Q$, the corresponding symplectic change of coordinates in $X = T^*Q$ is given by

$$(\tilde q, \tilde p) = (\Phi(q), [{}^t d\Phi(q)]^{-1}p)\,,$$

while the change of metric is given by

$$\tilde g(\tilde q) = [d\Phi(q)]^{-1,T}\, g(q)\, [d\Phi(q)]^{-1}\quad,\quad (\tilde g(\tilde q))^{-1} = [d\Phi(q)]g(q)^{-1}[d\Phi(q)]^T\,.$$

The function $\mathcal{E}(q,p) = |p|_q^2/2$ and the Hamiltonian vector field, $\mathcal{Y}_\mathcal{E}$, on T^*Q do not depend on the choice of coordinates on Q.

The manifold X is endowed with the riemannian metric $g^X = g \oplus^\perp g^{-1}$ where the orthogonal decomposition follows the decomposition into the horizontal and vertical spaces of $T_xX = (T_xX)^H \oplus (T_xX)^V$ with $x = (q,p)$ $(T_xX)^H \sim T_qQ$ and $(T_xX)^V \sim T_q^*Q$. More precisely at a point $x = (q,p)$, $g^X(x)$ is determined by

$$g^X(x; e_i, e_j) = g_{ij}(q) \quad , \quad g^X(x; \partial_{p_i}, \partial_{p_j}) = g^{ij}(q) \quad \text{and} \quad g^X(x; e_i, \partial_{p_j}) = 0\,.$$

Remark 6.1. *One may be puzzled by the sign in the expression $e_j = \partial_{q^j} + \Gamma_{ij}^\ell p_\ell \partial_{p_j}$ which does not coincide with the general definition of horizontal vector fields on a vector bundle $\pi : F \to Q$ endowed with an affine connection ∇^F. Actually we follow the convention of G. Lebeau and the horizontal tangent vectors on the cotangent bundle, $X = T^*Q$, are expressed with the Levi-Civita connection on the tangent bundle TQ formulated with the Christoffel symbols Γ_{ij}^ℓ. Compare in particular the adjoint formula* (58) *with* (57)*.*

The vertical Laplacian is given by $\Delta_p = \partial_{p_i} g_{ij}(q) \partial_{p_j} = \partial_p^T g(q) \partial_p$, which has also an invariant meaning.

Definition 6.2. *The geometric Kramers-Fokker-Planck operator on $X = T^*Q$ is the differential operator*

$$P_{\pm,Q,g} = \pm \mathcal{Y}_{\mathcal{E}} + \frac{1}{2}\left[-\Delta_p + |p|_q^2\right] .$$

The notation $\mathcal{O}_{Q,g}$ will be used for the operator acting along the fiber T_q^*Q:

$$\frac{1}{2}\left[-\Delta_p + |p|_q^2\right] = \frac{1}{2}\left[-\partial_p^T g(q) \partial_p + p^T g^{-1}(q) p\right] .$$

Note that the vertical operator $\mathcal{O}_{Q,g}$ is self-adjoint with the domain $D(\mathcal{O}_{Q,g}) = \{u \in L^2(X)\,,\, \mathcal{O}_{Q,g} u \in L^2(X)\}$ and it satisfies the uniform lower bound $\mathcal{O}_{Q,g} \geq \frac{\dim Q}{2}$.
We recall the notations introduced after (1) in the introduction.

Definition 6.3. *Let $(\overline{Q}, g)$, $\overline{Q} = Q \sqcup \partial Q$, be a riemannian manifold with a possibly non empty boundary ∂Q.*
For any $s' \in \mathbb{R}$, the space

$$\mathcal{H}^{s'}(q) = \left\{ u \in \mathcal{S}'(\mathbb{R}^d, dp)\,, \quad (\frac{d}{2} + \mathcal{O}_{g(q)})^{s'/2} u \in L^2(\mathbb{R}^d, dp) \right\}, \quad q \in Q\,,$$

defines a hermitian bundle $\pi_{\mathcal{H}^{s'}} : \mathcal{H}^{s'} \to \overline{Q}$. The space of its L^2-sections (resp. H^s-sections, $s \in \mathbb{R}$) is denoted by $L^2(Q;\mathcal{H}^{s'})$ (resp. $H^s(\overline{Q};\mathcal{H}^{s'})$).
On $L^2(Q;\mathcal{H}^{s'})$ the scalar product is given by

$$\langle u\,,\,v\rangle_{L^2(Q;\mathcal{H}^{s'})} = \langle u\,,\,(\frac{d}{2}+\mathcal{O}_g)^{s'/2}v\rangle$$

When Q is compact without boundary, the intersection $\cap_{s,s'\in\mathbb{R}} H^s(Q;\mathcal{H}^{s'})$ is denoted by $\mathcal{S}(X)$ and endowed with its Fréchet space topology. Its dual is $\mathcal{S}'(X)$.

When Q is compact without boundary, the space $\mathcal{S}(X)$ is the space of rapidly decaying $\mathcal{C}^\infty$-functions (like in $\mathbb{R}^{2d}$) and one easily checks (use for example the dyadic partition of unity in the p-variable recalled in Subsection 6.3.2):

$$\begin{aligned}
&\mathcal{S}'(X) = \mathop{\cup}_{s,s'\in\mathbb{R}} H^s(Q;\mathcal{H}^{s'})\,,\\
&\mathcal{C}_0^\infty(X) \subset \mathcal{S}(X) \subset H^{s_1}(Q;\mathcal{H}^{s_1'}) \subset H^{s_2}(Q;\mathcal{H}^{s_2'}) \subset \mathcal{S}'(X) \subset \mathcal{D}'(X)\,,
\end{aligned}$$

when $s_1 \geq s_2\,, s_1' \geq s_2'$, with dense and continuous embeddings.
We close this section by giving the explicit form of $P_{\pm,Q,g}$ in a specific local coordinate system along an hypersurface Q' of $\overline{Q}$ (which can be the boundary). In a neighborhood $\mathcal{V}_{q_0}$ of $q_0 \in Q'$ small enough, one can find a coordinate system (q^1, q') such that

$$\begin{aligned}
&q^1(q_0) = 0 \quad , \quad q'(q_0) = 0\,,\\
&\forall q \in \mathcal{V}_{q_0}\,, \quad (q \in Q') \Leftrightarrow \left(q^1(q) = 0\right)\,,\\
&\forall q \in \mathcal{V}_{q_0}\,, \quad g(q) = \begin{pmatrix} 1 & 0\\ 0 & m_{ij}(q^1,q')\end{pmatrix}\,.
\end{aligned}$$

In such a coordinate system the equalities

$$\begin{aligned}
2\mathcal{E} &= |p|_q^2 = |p_1|^2 + m^{ij}(q^1,q')p_i'p_j'\,,\\
\Delta_p &= \partial_{p_1}^2 + \partial_{p_i'} m_{ij}(q^1,q')\partial_{p_j'}\,,\\
\mathcal{Y}_{\mathcal{E}} &= p_1\partial_{q^1} + m^{ij}(q^1,q')p_i'\partial_{q'^j} - \frac{1}{2}[\partial_{q'^k} m^{ij}(q^1,q')]p_i'p_j'\partial_{p_k'}\\
&\qquad -\frac{1}{2}[\partial_{q^1} m^{ij}(q^1,q')]p_i'p_j'\partial_{p_1}\,,
\end{aligned}$$

hold with the convention $p'^T = (p_2, \ldots, p_d)$, $q'^T = (q^2, \ldots, q^d)$, and the corresponding summation rule. One obtains

$$P_{\pm,Q,g} = \pm p_1 \partial_{q^1} - \partial_{p_1}^2 + |p_1|^2 + P_{\pm,Q',m(q^1,.)} - \frac{1}{2}\partial_{q^1} m^{ij}(q^1,q')p'_i p'_j \partial_{p_1}\,. \quad (59)$$

This has some simple interpretation: when Q' is a curved hypersurface of Q, which can itself have some intrinsic curvature, the block component $(m_{ij}(q^1,q'))_{2\leq i,j\leq d}$ cannot be made independent of q_1. In particular the curvature of Q' has two consequences on the dynamics, a centrifugal (or centripetal) force and a Coriolis type force. The Coriolis force is a tangential phenomenon which can be included in $P_{\pm,Q',g(q^1,.)}$. Only the centrifugal force $\partial_{q^1} g^{ij}(q^1,q')p_i p_j \partial_{p_1}$ and the parametrization of the metric $m(q^1,.)$ by q^1, prevent from a separation of variables. This is why, even when $\overline{Q}$ is a domain of the flat euclidean space, the curvature term has to be analyzed accurately for boundary value problems. Even in this basic case, it is useful to formulate the problem in a geometric setting. We shall refer to the recent results by G. Lebeau in [Leb1][Leb2] and Bismut-Lebeau in [BiLe] where the problems raised by the curvature term have been accurately analyzed.

6.2 The result by G. Lebeau

In [Leb1][Leb2], G. Lebeau noticed that in local coordinates, the differential operators involved in $P_{\pm,Q,g} - z = \pm\mathcal{Y}_{\mathcal{E}} + \mathcal{O}_{Q,g} - \mathbb{R}\mathrm{e}\, z + i\mathrm{Im}\, z$ have a non trivial homogeneity: The main reason is that a symplectic change of variable $(q',p') = (\phi(q), d\phi(q)^{-T}p)$, $A^{-T} = (A^{-1})^T$, leads to $dq' = d\phi(q)d_q$, $dp' = (d\phi)^{-T}dp + (d(d\phi^{-T})p)dq$ and

$$\begin{aligned}
&\partial_{p'} = d\phi(q)\partial_p \\
&\partial_{q'} = (d\phi(q))^{-T}\left[\partial_q + (d(d\phi^T))(d\phi)^{-T}p)^T\partial_p\right]\,.
\end{aligned}$$

Hence assuming the order 1 for ∂_q in a invariant way imposes the order 1/2 for p and ∂_p.
Below is the list of orders for useful operators in a coordinate system:

$$\begin{aligned}
&\partial_{q^j} \ :\ 1 \quad ; \quad p_i, \partial_{p_i}\ :\ \frac{1}{2} \quad ; \quad e_j = \partial_{q^j} + \Gamma_{ij}^{\ell} p_\ell \partial_{p_i}\ :\ 1\,, \\
&\mathcal{Y}_{\mathcal{E}} = g^{ij}(q)p_i e_j\,,\ \mathrm{Im}\, z\ :\ \frac{3}{2} \quad ; \quad \mathcal{O}_{Q,g} = \frac{-\Delta_p + |p|_q^2}{2}\,,\ \mathbb{R}\mathrm{e}\ z\ :\ 1\,, \\
&\langle p\rangle\partial_{p_i}\,,\ e_j\,,\ \langle p\rangle^2 + |\,\mathbb{R}\mathrm{e}\ z| + \frac{|\mathrm{Im}\, z|}{\langle p\rangle}\ :\ 1\,, \quad (\langle p\rangle^2 = 1 + |p|_q^2)\,.
\end{aligned}$$

Definition 6.4. *Assume that (Q,g) is compact or $Q=\mathbb{R}^d$ with $g-Id_{\mathbb{R}^d}\in\mathcal{C}_0^\infty(\mathbb{R}^d)$. Consider a finite atlas $Q=\cup_\ell^L\Omega_\ell$ with $\overline{\Omega_\ell}$ compact when $\ell>1$ and let $\sum_{\ell=1}^L\theta_\ell(q)\equiv 1$, be a partition of unity subordinate to this atlas.*
With the corresponding coordinate system $(q^i,p_i)_{i=1,\ldots,d}$ on each $T^*\Omega_\ell$, $\ell\in\{1,\ldots,L\}$, set

$$\mathfrak{T}_\ell=\left\{\langle p\rangle\partial_{p_i}\ ,\ e_j\ ,\ \langle p\rangle^2+|\,\mathbb{R}\mathrm{e}\ z|+\frac{|Im\,z|}{\langle p\rangle}\right\}.$$

For $n\in\mathbb{N}$ and $z\in\mathbb{C}$, the spaces $\mathcal{W}^n_{g,z}$ are defined by induction according to

$$\begin{aligned}&\mathcal{W}^0_{g,z}=L^2(X,dqdp)=L^2(T^*Q,dqdp)\,,\\&(u\in\mathcal{W}^{n+1}_{g,z})\Leftrightarrow\left(\forall\ell\in\{1,\ldots,L\}\,,\forall T\in\mathfrak{T}_\ell\,,\ T[\theta_\ell(q)u]\in\mathcal{W}^n_{g,z}\right)\,,\\&\|u\|^2_{\mathcal{W}^{n+1}_{g,z}}=\sum_{\ell=1}^M\sum_{T\in\mathfrak{T}_\ell}\|T[\theta_\ell(q)u]\|^2_{\mathcal{W}^n_{g,z}}\,.\end{aligned}$$

The space $\mathcal{W}^s_{g,z}$ for $s\in\mathbb{R}$ are then defined by duality and interpolation.

For a fixed $s\in\mathbb{R}$, the space $\mathcal{W}^s_{g,z}$ does not depend on $z\in\mathbb{C}$ but its norm $\|\ \|_{\mathcal{W}^s_{g,z}}$ does. The quantities $\sum_{i=1}^d\|\langle p\rangle\partial_{p_i}v\|^2_{\mathcal{W}^n_{g,z}}+\|e_iv\|^2_{\mathcal{W}^n_{g,z}}$ and $\sum_{i=1}^d\|\langle p\rangle\partial_{p_i}v\|^2_{\mathcal{W}^n_{g,z}}+\|\partial_{q^i}v\|^2_{\mathcal{W}^n_{g,z}}$ are equivalent but the covariant derivative e_i is a more natural object. Note in particular the relations

$$\begin{aligned}&[e_k\,,|p|_q^2]=0\quad,\quad[e_k\,,\,e_{k'}]={}^tR_{kk'}(q)\in\mathrm{End}(T_q^*Q)\,,\\&[e_k\,,\,\Delta_p]=(\partial_{q^k}g_{ij})\partial_{p_i}\partial_{p_j}+\Gamma^\alpha_{\beta,k}g_{ij}\left[p_\alpha\partial_{p_\beta}\,,\,\partial_{p_i}\partial_{p_j}\right]\\&\qquad\qquad=\left(\partial_{q^k}g_{ij}-\Gamma^\alpha_{ik}g_{\alpha j}-\Gamma^\alpha_{jk}g_{i\alpha}\right)\partial_{p_i}\partial_{p_j}=0\,,\\\text{owing to}\quad&\Gamma^\alpha_{ik}g_{\alpha j}+\Gamma^\alpha_{jk}g_{i\alpha}=\partial_{q^k}g_{ij}\,.\end{aligned}$$

By introducing a dyadic partition of unity in the p-variable (see[Leb1][Leb2] or Subsection 6.3.2), the squared norm $\|u\|^2_{\mathcal{W}^s_{g,z}}$ is expressed as a quadratic series of parameter dependent usual Sobolev norms. We leave the reader to check with this approach that the space $\mathcal{W}^n_{g,z}$ does not depend on the the atlas and the subordinate partition of unity. Actually in [Leb1][Leb2] those spaces are defined with some parameter dependent pseudo-differential calculus on compact manifolds after introducing a dyadic partition of unity in the momentum variable p. Such a definition was extended in [Bisorb] for some locally symmetric non compact space Q. We focus here on the compact case and will later simply extend some estimates to the case of cylinders $\mathbb{R}\times Q$. Again because Q is compact without boundary, one has

$$
\begin{aligned}
&\mathcal{S}(X)=\mathop{\cap}_{s\in\mathbb{R}}\mathcal{W}^{s}_{g,z} \quad , \quad \mathcal{S}'(X)=\mathop{\cup}_{s\in\mathbb{R}}\mathcal{W}^{s}_{g,z}\,,\\
&\mathcal{C}^{\infty}_{0}(X)\subset\mathcal{S}(X)\subset\mathcal{W}^{s_1}_{g,z}\subset\mathcal{W}^{s_2}_{g,z}\subset\mathcal{S}'(X)\subset\mathcal{D}'(X)\,,\quad s_1\geq s_2\,,
\end{aligned}
$$

with dense and continuous embeddings.

Theorem 1.2 of [Leb2]. *The riemannian manifold (Q,g) is assumed to be compact (without boundary). There exists $\delta_1>0$ such that the following properties hold.*
As an operator in $\mathcal{S}(X)$ or $\mathcal{S}'(X)$, $P_{\pm,Q,g}$ satisfies

$$
\sigma(P_{\pm,Q,g})\subset\left\{z\in\mathbb{C},\mathbb{R}\mathrm{e}\;z\geq\frac{dim\;(Q)}{2}+\delta_1|Im\,z|^{1/2}\right\}\,.
$$

For all $u\in\mathcal{S}'(X)$ and all $z\in U_{\delta_1}=\left\{z\in\mathbb{C}\,,\mathbb{R}\mathrm{e}\;z<\delta_1|Im\,z|^{1/2}\right\}$, the condition $(P_{\pm,Q,g}-z)u\in\mathcal{W}^{s}_{g,0}$, $s\in\mathbb{R}$, implies $u\in\mathcal{W}^{s+\frac{2}{3}}_{g,0}$, $\mathcal{O}_g u\in\mathcal{W}^{s}_{g,z}$, $\mathcal{Y}_{\mathcal{E}}u\in\mathcal{W}^{s}_{g,z}$ with the estimate

$$
\begin{aligned}
C_s\|(P_{\pm,Q,g}-z)u\|_{\mathcal{W}^{s}_{g,z}}\geq{}&\|\mathcal{O}_{Q,g}u\|_{\mathcal{W}^{s}_{g,z}}+\|(\pm\mathcal{Y}_{\mathcal{E}}-iIm\,z)u\|_{\mathcal{W}^{s}_{g,z}}\\
&+(|\mathbb{R}\mathrm{e}\;z|+|Im\,z|^{1/2})\|u\|_{\mathcal{W}^{s}_{g,z}}+\|u\|_{\mathcal{W}^{s+\frac{2}{3}}_{g,z}}
\end{aligned}
\tag{60}
$$

for some constant C_s.

We shall only use those estimates with $s\in[-1,1]$ and make the connection with more usual Sobolev estimates. Note in particular for $s\in[0,1]$ the embeddings and norm estimates

$$
\begin{aligned}
&\mathcal{W}^{s}_{g,z}\subset\mathcal{W}^{s}_{g,0}\subset H^{s}(Q;\mathcal{H}^0)\,,\\
&\frac{1}{C_s}\|u\|_{H^{s}(Q;\mathcal{H}^0)}\leq\|u\|_{\mathcal{W}^{s}_{g,0}}\leq\|u\|_{\mathcal{W}^{s}_{g,z}}\,,\\
&H^{-s}(Q;\mathcal{H}^0)\subset\mathcal{W}^{-s}_{g,0}\subset\mathcal{W}^{-s}_{g,z}\,,\\
&\|u\|_{\mathcal{W}^{-s}_{g,z}}\leq\|u\|_{\mathcal{W}^{-s}_{g,0}}\leq C_s\|u\|_{H^{-s}(Q;\mathcal{H}^0)}\,,
\end{aligned}
$$

obtained by duality and interpolation from the obvious cases $s=0$ and $s=1$.

Corollary 6.5. *Assume that (Q,g) is a compact riemannian manifold (without boundary). Consider the operator $K_{\pm,Q,g}-\frac{dim\,Q}{2}$ defined in $L^2(X)=L^2(T^*Q,dqdp)$ by*

$$D(K_{\pm,Q,g})=\{u\in L^2(Q;\mathcal{H}^1)\,,\ P_{\pm,Q,g}u\in L^2(X)\}$$
$$\forall u\in D(K_{\pm,Q,g})\,,\quad K_{\pm,Q,g}u=P_{\pm,Q,g}u=(\pm\mathcal{Y}_{\mathcal{E}}+\mathcal{O}_{Q,g})u\,,$$

is maximal accretive and its adjoint is given by $K_{\pm,Q,g}^*=K_{\mp,Q,g}$.
The domains $D(K_{\pm,Q,g})$ and $D(K_{\mp,Q,g})=D(K_{\pm,Q,g}^*)$ are equal with equivalent graph norms. The spaces $\mathcal{S}(X)$ and $\mathcal{C}_0^\infty(X)$ are dense in $D(K_{\pm,Q,g})$ endowed with its graph norm.
The estimates

$$\|\mathcal{O}_{Q,g}u\|+\|(\pm\mathcal{Y}_{\mathcal{E}}-i\lambda)u\|+\langle\lambda\rangle^{\frac{1}{2}}\|u\|+\|u\|_{H^{2/3}(Q;\mathcal{H}^0)}\leq C\|(K_{\pm,Q,g}-i\lambda)u\|\,,\tag{61}$$

$$\frac{dim\,Q}{2}\|u\|^2\leq\|u\|^2_{L^2(Q;\mathcal{H}^1)}\leq\mathbb{R}\mathrm{e}\,\langle u\,,\,(K_{\pm,Q,g}-i\lambda)u\rangle\tag{62}$$

holds all $u\in D(K_{\pm,Q,g})$ and all $\lambda\in\mathbb{R}$.
Moreover for any $\lambda\in\mathbb{R}$, the resolvent $(K_{\pm,Q,g}-i\lambda)^{-1}$ extends by continuity

- *as an element of $\mathcal{L}(L^2(Q;\mathcal{H}^{-1});L^2(Q;\mathcal{H}^1))$ with*

$$\|(K_{\pm,Q,g}-i\lambda)^{-1}u\|_{L^2(Q;\mathcal{H}^1)}\leq\|u\|_{L^2(Q;\mathcal{H}^{-1})}\,,$$

- *and as an element of $\mathcal{L}(H^s(Q;\mathcal{H}^0);H^{s+\frac{2}{3}}(Q;\mathcal{H}^0))$ for any $s\in[-\frac{2}{3},0]$ with*

$$\|(K_{\pm,Q,g}-i\lambda)^{-1}u\|_{H^{s+\frac{2}{3}}(Q;\mathcal{H}^0)}\leq C_s\|u\|_{H^s(Q;\mathcal{H}^0)}\,.$$

Proof. The operator $P_{\pm,Q,g}-\frac{\dim Q}{2}$ initially defined with the domain $\mathcal{S}(X)$, is accretive owing to the simple integration by parts

$$\forall u\in\mathcal{S}(X)\,,\quad\mathbb{R}\mathrm{e}\,\langle u\,,\,P_{\pm,Q,g}u\rangle=\|u\|^2_{L^2(Q;\mathcal{H}^1)}\geq\frac{\dim Q}{2}\|u\|^2\,.$$

According to [ReSi75], its closure denoted by $K_{\pm,Q,g}=\overline{P_{\pm,Q,g}}$ is accretive. It is actually maximal accretive because the equation $P_{\mp,Q,g}=P_{\pm,Q,g}^*u=0$ in $\mathcal{S}'(X)$ and the regularity result of Theorem 1.2 in [Leb2] implies $u\in\mathcal{S}(X)$ and $u=0$. Therefore the range $\mathrm{Ran}\,P_{\pm,Q,g}$ is dense in $L^2(X)$ (use

$\overline{\mathrm{Ran}\, P_{\pm,Q,g}} = \ker(P^*_{\pm,Q,g}) = \{0\})$. This suffices to prove the maximal accretivity of $K_{\pm,Q,g}$.
From this definition of $K_{\pm,Q,g}$, $\mathcal{S}(X)$ (and therefore $\mathcal{C}_0^\infty(X)$ because $P_{\pm,Q,g}$ has polynomially controlled coefficients) is a core for $K_{\pm,Q,g}$. The subelliptic estimates, the identification of the domain $D(K_{\pm,Q,g})$ as it is stated, the equality $D(K_{\pm,Q,g}) = D(K_{\mp,Q,g}) = D(K^*_{\pm,Q,g})$ are direct consequences of (60) applied with $s = 0$. We use the upper bound

$$\|u\|_{H^{\frac{2}{3}}(Q;\mathcal{H}^0)} \leq C\|u\|_{\mathcal{W}^{\frac{2}{3}}_{g,i\lambda}} .$$

The extension of $(K_{\pm,Q,g} - i\lambda)^{-1} \in \mathcal{L}(L^2(Q;\mathcal{H}^{-1}); L^2(Q;\mathcal{H}^{-1}))$ is contained in (62).
The extension of $(K_{\pm,Q,g} - i\lambda)^{-1} \in \mathcal{L}(H^s(Q;\mathcal{H}^0); H^{s+\frac{2}{3}}(Q;\mathcal{H}^0))$ comes from

$$\|u\|_{H^{s+\frac{2}{3}}(Q;\mathcal{H}^0)} \leq C_s\|u\|_{\mathcal{W}^{s+\frac{2}{3}}_{g,i\lambda}} \leq C'_s\|u\|_{\mathcal{W}^s} \leq C''_s\|u\|_{H^s(Q;\mathcal{H}^0)} ,$$

due to (60) with $s \leq 0$ and $s + \frac{2}{3} \geq 0$. $\square$

6.3 Partitions of unity

Like in [Leb1][Leb2], we introduce two partitions of unity: a spatial partition of unity which allows to use coordinate systems, a dyadic partition of unity in the momentum variable which replaces the analysis as $p \to \infty$ by some parameter dependent problem. In [Leb1][Leb2], all formulas are first proved for elements of $\mathcal{S}(X)$ and extended by density arguments. Here, in order to extend those decompositions to boundary value problems where a possibly non regularity-preserving operator A prevents from identifying a core of regular functions, various quantities are directly calculated for $u \in L^2(Q;\mathcal{H}^1)$ such that $P_{\pm,Q,g}u \in L^2(Q;\mathcal{H}^{-1})$ by using the $L^2(X)$ scalar product and the $L^2(Q;\mathcal{H}^1) - L^2(Q;\mathcal{H}^{-1})$ duality product (both denoted by $\langle\ ,\ \rangle$).

6.3.1 Spatial partition of unity

The riemannian manifold $(\overline{Q} = Q \sqcup \partial Q, g)$ can be

- a compact manifold without or with a boundary,
- a compact perturbation of the euclidean space $\mathbb{R}^d$ or the half-space $\overline{\mathbb{R}^d}_- = (-\infty, 0] \times \mathbb{R}^{d-1}$ with $g_{ij} - \delta_{ij} \in \mathcal{C}_0^\infty(\mathbb{R}^d)$,

- a cylinder $\mathbb{R} \times Q'$ or a half cylinder $(-\infty, 0] \times Q'$ with Q' compact, $\partial Q' = \emptyset$, $g = 1 \oplus m(q^1, q')$, $m_{ij} - m_{ij}(-\infty, q') \in \mathcal{C}_0^\infty(\overline{Q})$.

Such a manifold $\overline{Q}$ admits finite and countable locally finite coverings $\overline{Q} = \cup_{\ell \in \mathcal{L}} \Omega_\ell$ where Ω_ℓ is diffeomorphic to a regular domain of $\mathbb{R}^d$ and $g_{ij}(q) \in \mathcal{C}_b^\infty(\Omega_\ell)$ in the corresponding coordinate system, with a bounded intersection number

$$\sup_{\ell' \in \mathcal{L}} \sharp \left\{\ell \in \mathcal{L},\ \Omega_\ell \cap \Omega_{\ell'} \neq \emptyset\right\} \leq N_\Omega \in \mathbb{N}.$$

Furthermore the covering can be chosen such that there exists a partition of unity $\chi = (\chi_\ell)_{\ell \in \mathcal{L}}$, $\chi_\ell \in \mathcal{C}_b^\infty(\Omega_\ell)$ with uniform seminorms w.r.t $\ell \in \mathcal{L}$, $\operatorname{supp} \chi_\ell \subset \Omega_\ell$ and $\sum_{\ell \in \mathcal{L}} \chi_\ell^2 = 1$. The simplest way consists in taking $\mathcal{L}$ finite. Due to the vertical nature of $\mathcal{O}_{Q,g}$, the condition $u \in L^2(Q; \mathcal{H}^{s'})$ is equivalent to

$$\left\{ \begin{array}{l} \forall \ell \in \mathcal{L}, \quad \chi_\ell u \in L^2(\Omega_\ell; \mathcal{H}^{s'}), \\ \sum_{\ell \in \mathcal{L}} \|\chi_\ell u\|^2_{L^2(\Omega_\ell; \mathcal{H}^{s'})} < +\infty, \end{array} \right.$$

with the equality of squared norms $\|u\|^2_{L^2(Q;\mathcal{H}^{s'})} = \sum_{\ell \in \mathcal{L}} \|\chi_\ell u\|^2_{L^2(\Omega_\ell;\mathcal{H}^{s'})}$. We recall also (see for instance [ChPi]), that when $\partial Q = \emptyset$ the usual Sobolev norms in q defined by $\|u\|_{H^s(\overline{Q};\mathcal{H}^0)} = \|(1 - \Delta_q)^{s/2} u\|$ with $\Delta_q = \partial_{q^i} g^{ij}(q) \partial_{q^j}$ are equivalent via a change of the metric and satisfy

$$\left(\frac{\sum_{\ell \in \mathcal{L}} \|\chi_\ell u\|^2_{H^s(Q;\mathcal{H}^0)}}{\|u\|^2_{H^s(Q;\mathcal{H}^0)}} \right)^{\pm 1} \leq C_{s,g,\chi}.$$

In the case when $\partial Q \neq \emptyset$, Sobolev spaces are defined by using locally the reflection principle (see e.g. [ChPi]).

Proposition 6.6. *Take the above partition of unity and assume $z \in \mathbb{C}$, $s' \in [-1, 0]$. For $u \in L^2(Q; \mathcal{H}^1)$, the condition $(P_{\pm,Q,g} - z)u \in L^2(Q; \mathcal{H}^{s'})$ is equivalent to*

$$\forall \ell \in \mathcal{L}, \quad P_{\pm,Q,g}(\chi_\ell u) \in L^2(Q; \mathcal{H}^{s'}). \tag{63}$$

There exists a constant $C_\chi > 0$ such that

$$\left(\frac{\sum_{\ell \in \mathcal{L}} \|(P_{\pm,Q,g} - z)\chi_\ell u\|^2_{L^2(Q;\mathcal{H}^{s'})} + \|\chi_\ell u\|^2_{L^2(Q;\mathcal{H}^1)}}{\|(P_{\pm,Q,g} - z)u\|^2_{L^2(Q;\mathcal{H}^{s'})} + \|u\|^2_{L^2(Q;\mathcal{H}^1)}} \right)^{\pm 1} \leq C_\chi, \tag{64}$$

holds uniformly with respect to (u,z,s') *.*
Two functions $u_1, u_2 \in L^2(Q;\mathcal{H}^1)$ *such that* $P_{\pm,Q,g}u_1, P_{\pm,Q,g}u_2 \in L^2(Q;\mathcal{H}^{-1})$ *, satisfy the identity*

$$\langle u_1\,,\,P_{\pm,Q,g}u_2\rangle = \sum_{\ell\in\mathcal{L}}\langle \chi_\ell u_1\,,\,P_{\pm,Q,g}\chi_\ell u_2\rangle\,. \tag{65}$$

Proof. When $u\in L^2(Q;\mathcal{H}^1)$ satisfies $(P_{\pm,Q,g}-z)u\in L^2(Q;\mathcal{H}^{s'})$, (63) comes from

$$(P_{\pm,Q,g}-z)\chi_\ell u = \chi_\ell(P_{\pm,Q,g}-z)u \pm g^{ij}(q)(\partial_{q_j}\chi_\ell(q))p_j u\,.$$

With $s'\leq 0$, it implies

$$\begin{aligned}\sum_{\ell\in\mathcal{L}}\|(P_{\pm,Q,g}-z)\chi_\ell u\|^2_{L^2(Q;\mathcal{H}^{s'})} &\leq 2\sum_{\ell\in\mathcal{L}}\|\chi_\ell(P_{\pm,Q,g}-z)u\|^2_{L^2(Q;\mathcal{H}^{s'})}\\ &\qquad\qquad +\|g^{ij}(q)(\partial_{q_j}\chi_\ell(q))p_j u\|^2\\ &\leq C^1_\chi\left[\|(P_{\pm,Q,g}-z)u\|^2_{L^2(Q;\mathcal{H}^{s'})}+\|u\|^2_{L^2(Q;\mathcal{H}^1)}\right]\,,\end{aligned}$$

because $\sum_{\ell'\in\mathcal{L}}|\partial_q\chi_\ell(q)|^2\leq N_\Omega\sup_{\ell\in\mathcal{L}}\|\partial_q\chi_\ell\|^2_{L^\infty}<\infty$.
The relation (131) applied with $P=P_{\pm,Q,g}$ and $\mathrm{ad}^{\,2}_{\chi_\ell}P_{\pm,Q,g}=0$, gives here

$$P_{\pm,Q,g}=\sum_{\ell\in\mathcal{L}}\chi_\ell P_{\pm,Q,g}\chi_\ell\,.$$

Hence (63) implies $\chi_{\ell'}(P_{\pm,Q,g}-z)u\in L^2(Q;\mathcal{H}^{s'})$ for $\ell'\in\mathcal{L}$, with the estimate

$$\begin{aligned}\|\chi_{\ell'}(P_{\pm,Q,g}-z)u\|^2_{L^2(Q;\mathcal{H}^{s'})} &\leq \|\sum_{\ell\in\mathcal{L}}\chi_{\ell'}\chi_\ell P_{\pm,Q,g}\chi_\ell u\|^2_{L^2(Q;\mathcal{H}^{s'})}\\ &\leq N_\Omega\sum_{\chi_\ell\chi_{\ell'}\neq 0}\|(P_{\pm,Q,g}-i\lambda)\chi_\ell u\|^2_{L^2(Q;\mathcal{H}^{s'})}\,.\end{aligned}$$

Taking the sum over $\ell'\in\mathcal{L}$ yields

$$\|(P_{\pm,Q,g}-i\lambda)u\|^2_{L^2(Q,\mathcal{H}^{s'})}\leq N_\Omega^2\sum_{\ell\in\mathcal{L}}\|(P_{\pm,Q,g}-i\lambda)\chi_\ell u\|^2_{L^2(Q;\mathcal{H}^{s'})}\,.$$

The above identity also proves(65). $\square$

6.3.2 Dyadic momentum partition of unity

The previous partition of unity in the q-variable allows to focus on the case when $\overline{Q}$ is endowed with a global coordinate system, namely:

- $\overline{Q}$ is a compact perturbation of the euclidean space $\mathbb{R}^d$ or half-space $\overline{\mathbb{R}^d}_- = (-\infty, 0] \times \mathbb{R}^{d-1}$,
- $\overline{Q}$ is the torus $\mathbb{T}^d$, the cylinder $\mathbb{R}\times\mathbb{T}^{d-1}$ or half-cylinder $(-\infty,0]\times\mathbb{T}^{d-1}$ with the metric $1\oplus m(q^1,q')$, $m_{ij} - m_{ij}(-\infty, q') \in \mathcal{C}_0^\infty(\overline{Q})$.

Take two functions $\tilde{\chi}_0 \in \mathcal{C}_0^\infty(\mathbb{R})$ and $\tilde{\chi}_1 \in \mathcal{C}_0^\infty((0,+\infty))$ such that $\tilde{\chi}_0^2(t) + \sum_{\ell\in\mathbb{N}^*}\tilde{\chi}_1^2(2^{-\ell}t) \equiv 1$ on $[0,+\infty)$. Set $\chi_\ell(q,p) = \tilde{\chi}_1(2^{-\ell}|p|_q)$ for $\ell \in \mathbb{N}^*$, $\chi_0(q,p) = \tilde{\chi}_0(|p|_q)$ and $\chi = (\chi_\ell)_{\ell\in\mathcal{L}}$, $\mathcal{L} = \mathbb{N}$.
The two following lemmas recall the equivalence of norms for the functional spaces $L^2(Q;\mathcal{H}^{s'})$ and $H^s(Q;\mathcal{H}^0)$. A proposition similar to Proposition 6.6 is proved afterwards.

Lemma 6.7. *With the above partition of unity $\chi = (\chi_\ell)_{\ell\in\mathcal{L}}$, $\chi_\ell = \tilde{\chi}_{1,0}\left(\frac{|p|_q}{2^\ell}\right)$, $\sum_{\ell\in\mathcal{L}}\chi_\ell^2 \equiv 1$ and $\mathcal{L} = \mathbb{N}$, the norms $\left(\sum_{\ell\in\mathcal{L}}\|\chi_\ell u\|^2_{L^2(Q;\mathcal{H}^{s'})}\right)^{1/2}$ and $\|u\|_{L^2(Q;\mathcal{H}^{s'})}$ are equivalent for any $s' \in \mathbb{R}$.*

Proof. For the equivalence of the norms in $L^2(Q;\mathcal{H}^{s'})$, we can assume $Q = \mathbb{R}^d$, $g_{ij}(q) = \delta_{ij}$ and consider first $s' = n \in \mathbb{N}$. The result for a general $s' \in \mathbb{R}$ follows by duality and interpolation (see the application of Lemma B.1). Moreover density arguments are allowed here and all the calculations can be made with $u \in \mathcal{S}(X) = \mathcal{S}(\mathbb{R}^{2d})$.
We have to compare $\sum_{|\alpha|+m\leq n}\|\langle p\rangle^m\partial_p^\alpha u\|^2$ and $\sum_{\ell\in\mathbb{N},|\alpha|+m\leq n}\|\langle p\rangle^m\partial_p^\alpha\chi_\ell u\|^2$.
Introduce for $m \in \mathbb{N}$ and $m' \in \mathbb{R}$, the class $\mathrm{DiffP}^{m,m'}$ of differential operators $a(p,D_p) = \sum_{|\alpha|\leq m} a_\alpha(p)D_p^\alpha$

$$\forall\beta\in\mathbb{N}^{\dim Q}\,,\quad N^{m,m'}_{\alpha,\beta}(a) = \sup_{p\in\mathbb{R}^d}\frac{|\partial_p^\beta a_\alpha(p)|}{\langle p\rangle^{m'-|\beta|}} < +\infty\,.$$

For every $(m,m') \in \mathbb{N}\times\mathbb{R}$, $\mathrm{DiffP}^{m,m'}$ endowed with the seminorms $N^{m,m'}_{\alpha,\beta}$, $|\alpha| \leq m, \beta \in \mathbb{N}^{\dim Q}$ is a Fréchet space. The union $\mathrm{DiffP} = \cup_{(m,m')\in\mathbb{N}\times\mathbb{R}}\mathrm{DiffP}^{m,m'}$

is a bigraded algebra [2] such that

$$
\begin{aligned}
&\mathrm{DiffP}^{m_1,m_1'}\circ \mathrm{DiffP}^{m_2,m_2'}\subset \mathrm{DiffP}^{m_1+m_2,m_1'+m_2'}\,,\\
&\left[\mathrm{DiffP}^{m_1,m_1'}\,,\,\mathrm{DiffP}^{m_2,m_2'}\right]\subset \mathrm{DiffP}^{m_1+m_2-1,m_1'+m_2'-1}\,,
\end{aligned}
$$

the corresponding mappings $(A,B)\mapsto A\circ B$ and $(A,B)\mapsto [A,B]$ being bilinearly continuous. Note also that the p-support

$$
p-\mathrm{supp}\,a(p,D_p)=\underset{|\alpha|\leq m}{\cup}\mathrm{supp}\,a_\alpha\,,
$$

satifies

$$
p-\mathrm{supp}\,[a(p,D_p)\circ b(p,D_p)]\subset p-\mathrm{supp}\,a(p,D_p)\cap p-\mathrm{supp}\,b(p,D_p)\,.
$$

The χ_ℓ's, with $\chi_\ell(p)=\tilde{\chi}_{0,1}\left(\frac{|p|}{2^\ell}\right)$, make a uniformly bounded family of $\mathrm{DiffP}^{0,0}$ with $p-\mathrm{supp}\,\nabla\chi_\ell\subset\left\{\frac{2^\ell}{C}\leq |p|\leq C2^\ell\right\}$.
The proof of the equivalence of norms is now done by induction:

- For $n=0$ it is obvious: $L^2(Q;\mathcal{H}^0)=L^2(X)$.
- If it is true for all $n'\leq N\in\mathbb{N}$, apply the formula (130) with $P=\langle p\rangle^m D_p^\alpha$ and $|\alpha|+m=n+1$. It gives for any $u\in\mathcal{S}(X)=\mathcal{S}(\mathbb{R}^{2d})$,

  $$
  \|\langle p\rangle^m D_p^\alpha u\|^2-\sum_{\ell\in\mathcal{L}}\|\langle p\rangle^m D_p^\alpha\chi_\ell u\|^2=-\sum_{\ell\in\mathcal{L}}\|A_\ell u\|^2+\mathbb{R}\mathrm{e}\,\langle u\,,\,B_\ell u\rangle\,,
  $$

  where $A_\ell=\mathrm{ad}_{\chi_\ell}(\langle p\rangle^m D_p^\alpha)$ is uniformly bounded in $\mathrm{DiffP}^{(|\alpha|-1)_+,(m-1)_+}$, $B_\ell=D_p^\alpha\langle p\rangle^m\mathrm{ad}_{\chi_\ell}^2(\langle p\rangle^m D_p^\alpha)$ is uniformly bounded in $\mathrm{DiffP}^{(2|\alpha|-2)_+,(2m-2)_+}$, with $p-$supports contained in $\left\{\frac{2^\ell}{C}\leq |p|\leq C2^\ell\right\}$ and covered by a fixed number of $\chi_{\ell'}$. The equivalence of norms for $n'\leq n$ implies that

  $$
  \left|\|\langle p\rangle^m D_p^\alpha u\|^2-\sum_{\ell\in\mathcal{L}}\|\langle p\rangle^m D_p^\alpha\chi_\ell u\|^2\right|
  $$

[2] The aware reader will recognize a subalgebra of the Weyl-Hörmander pseudodifferential class associated with the metric $\frac{dp^2}{\langle p\rangle^2}+\frac{d\eta^2}{\langle\eta\rangle^2}$ and the gain function $\langle p\rangle\langle\eta\rangle$ (see [HormIII]-Chap 18 or [Hel1]). The restriction to differential operators, which contains differential operators with polynomial coefficients, allows an easier handling of supports without requiring the more sophisticated notion of confinement (see [BoLe]).

is estimated by

$$C\|u\|^2_{L^2(Q;\mathcal{H}^n)} \quad \text{or} \quad C'\sum_{\ell\in\mathcal{L}}\|\chi_\ell u\|^2_{L^2(Q;\mathcal{H}^n)}\,.$$

Since $|a_{n+1}-b_{n+1}| \leq C\sum_{p\leq n} b_p$ and $\sum_{p\leq n} a_p \leq C\sum_{p\leq n} b_b$ imply $\sum_{p\leq n+1} a_p \leq C'\sum_{p\leq n+1} b_p$, this proves the equivalence for $n'=n+1$. The density of $\mathcal{S}(X)=\mathcal{S}(\mathbb{R}^{2d})$ in $L^2(Q;\mathcal{H}^{n+1})$ ends the proof.

□

Lemma 6.8. *With the above partition of unity* $\chi=(\chi_\ell)_{\ell\in\mathcal{L}}$, $\chi_\ell=\tilde{\chi}_{1,0}\left(\frac{|p|_q}{2^\ell}\right)$, $\sum_{\ell\in\mathcal{L}}\chi_\ell^2\equiv 1$ *and* $\mathcal{L}=\mathbb{N}$. *For any* $s\in[0,1]$ *there exists* $C_{\chi,s}$ *such that*

$$\left(\frac{\sum_{\ell\in\mathcal{L}}\|\chi_\ell u\|^2_{H^s(\overline{Q};\mathcal{H}^0)}}{\|u\|^2_{H^s(\overline{Q};\mathcal{H}^0)}}\right)^{\pm 1} \leq C_{\chi,s}\,.$$

Proof. Again the spatial partition of unity of Subsection 6.3.1, the reflection principle and the equivalence of norms for two different metrics restrict the analysis to the case $Q=\mathbb{R}^d$, $g_{ij}=\delta_{ij}$.
The result is obvious when $s=0$, $H^0(\mathbb{R}^d;\mathcal{H}^0)=L^2(\mathbb{R}^{2d},dqdp)$.
Consider now the case $s=1$, with

$$\|u\|^2_{H^1(Q;\mathcal{H}^0)}=\|u\|^2+\sum_{j=1}^d\|\partial_{q^j}u\|^2\,.$$

We use again the formula (130)

$$\|\partial_{q^j}u\|^2-\sum_{\ell\in\mathcal{L}}\|\partial_{q^j}\chi_\ell u\|^2=-\sum_{\ell\in\mathcal{L}}\|(\mathrm{ad}_{\chi_\ell}\partial_{q^j})u\|^2\,,$$

with

$$\begin{aligned}\mathrm{ad}_{\chi_\ell}\partial_{q^j} &= (\partial_{q^j}\chi_\ell)=\frac{\partial_{q^j}(|p|_q)}{2^\ell}\tilde{\chi}'_{0,1}(\frac{|p|_q}{2^\ell})\\ \text{and}\qquad \mathrm{ad}^2_{\chi_\ell}\partial_{q^j} &= 0\,.\end{aligned}$$

The derivative $\partial_{q^j}|p|_q$ satisfies

$$\frac{1}{2^\ell}\partial_{q^j}|p|_q=\frac{\partial_{q^j}g^{ik}(q)p_ip_k}{2\times 2^\ell(g^{ik}(q)p_ip_k)^{1/2}}=\mathcal{O}(1) \quad \text{in } \operatorname{supp}\tilde{\chi}'_{0,1}\left(\frac{|p|_q}{2^\ell}\right)\,.$$

and this proves

$$\left| \|u\|^2_{H^1(\mathbb{R}^d;\mathcal{H}^0)} - \sum_{\ell\in\mathcal{L}} \|\chi_\ell u\|^2_{H^1(\mathbb{R}^d;\mathcal{H}^0)} \right| \leq C^1_\chi \|u\|^2\,,$$

which implies the equivalence of norms for $s=1$.
The general case follows by interpolation with $\|u\|_{H^r(\mathbb{R}^d;\mathcal{H}^0)} = \|(1-\Delta_q)^{r/2}u\|$ (see Lemma B.1 and its application). $\square$

While working with $P_{\pm,Q,g}$, we avoid again density arguments in order to allow the same line for general boundary value problems.

Proposition 6.9. *Take the above dyadic partition of unity* $\chi = (\chi_\ell)_{\ell\in\mathcal{L}}$, $\chi_\ell = \tilde{\chi}_{1,0}\left(\frac{|p|_q}{2^\ell}\right)$, $\sum_{\ell\in\mathcal{L}}\chi_\ell^2 \equiv 1$ *and* $\mathcal{L}=\mathbb{N}$, *and let* $u\in L^2(Q;\mathcal{H}^1)$, $z\in\mathbb{C}$ *and* $s'\in[-1,0]$.
The condition $(P_{\pm,Q,g}-z)u \in L^2(Q;\mathcal{H}^{s'})$ *is equivalent to* $\sum_{\ell\in\mathcal{L}}\|(P_{\pm,Q,g}-z)\chi_\ell u\|^2_{L^2(Q;\mathcal{H}^{s'})} < +\infty$, *and the equivalence of norms* (64) *still holds (with the new choice* $\chi_\ell=\chi_\ell(p)$*).*
For any pair $u_1,u_2\in L^2(Q;\mathcal{H}^1)$ *such that* $P_{\pm,Q,g}u_1, P_{\pm,Q,g}u_2 \in L^2(Q;\mathcal{H}^{-1})$, *the identity* (65) *is now replaced by*

$$\left|\langle u_1\,,\, P_{\pm,Q,g}u_2\rangle - \sum_{\ell\in\mathcal{L}}\langle \chi_\ell u_1\,,\, P_{\pm,Q,g}\chi_\ell u_2\rangle\right| \leq C_\chi\|u_1\|\|u_2\|\,. \tag{66}$$

Proof. The proof is essentially the same as for Proposition 6.6. We shall use the notation $\mathrm{DiffP}^{m,m'}$ and $p-\mathrm{supp}$ introduced in the proof of Lemma 6.7. With $[\mathcal{Y}_{\mathcal{E}},\chi_\ell] = (\mathcal{Y}_{\mathcal{E}}\chi_\ell)=0$, the commutator $\mathrm{ad}_{\chi_\ell}(P_{\pm,Q,g}-z)$ equals

$$\mathrm{ad}_{\chi_\ell}P_{\pm,Q,g} = \left[\chi_\ell, -\frac{\Delta_p}{2}\right] = (\nabla_p\chi_\ell).\nabla_p + \frac{1}{2}(\Delta_p\chi_\ell) \ \in \mathrm{DiffP}^{1,-1}\,,$$

where the gradients in the p-variable, the Laplace operator Δ_p and the scalar product denoted by $x.y$ are given by the scalar product $g(q)$. In particular $\chi_\ell(p) = \tilde{\chi}_{0,1}\left(\frac{|p|_q}{2^\ell}\right)$ gives:

$$\nabla_p\chi_\ell(p) = \frac{p}{2^\ell|p|_q}\tilde{\chi}'_{0,1}\left(\frac{|p|_q}{2^\ell}\right) \quad \text{and} \quad \Delta_p\chi_\ell(p) = \frac{1}{2^{2\ell}}\tilde{\chi}''_{0,1}\left(\frac{|p|_q}{2^\ell}\right)\,.$$

This also implies $p-\mathrm{supp}(\mathrm{ad}_{\chi_\ell} P_{\pm,Q,g})\subset\left\{\frac{2^\ell}{C}\leq |p|_q\leq C2^\ell\right\}$.
a) Assume $u\in L^2(Q;\mathcal{H}^1)$ and $(P_{\pm,Q,g}-z)u\in L^2(Q;\mathcal{H}^{s'})$ with $s'\in[-1,0]$. The equality

$$\chi_\ell(P_{\pm,Q,g}-z)u=(P_{\pm,Q,g}-z)\chi_\ell u+(\mathrm{ad}_{\chi_\ell} P_{\pm,Q,g})u$$

leads to

$$\begin{aligned}\sum_{\ell\in\mathcal{L}}\|(P_{\pm,Q,g}-z)\chi_\ell u\|^2_{L^2(Q;\mathcal{H}^{s'})} \;\leq\; & 2\sum_{\ell\in\mathcal{L}}\|\chi_\ell(P_{\pm,Q,g}-z)u\|^2_{L^2(Q;\mathcal{H}^{s'})}\\ & +2\sum_{\ell\in\mathcal{L}}\|(\mathrm{ad}_{\chi_\ell} P_{\pm,Q,g})u\|^2\,.\end{aligned}$$

Lemma 6.7 says that the first term is less than $C^1_\chi\|(P_{\pm,Q,g}-z)u\|^2_{L^2(Q;\mathcal{H}^{s'})}$. The listed properties of $\mathrm{ad}_{\chi_\ell}P_{\pm,Q,g}$ imply

$$\sum_{\ell\in\mathcal{L}}\|(\mathrm{ad}_{\chi_\ell} P_{\pm,Q,g})u\|^2\leq C^2_\chi\sum_{\ell\in\mathcal{L}}\|\sum_{\chi_\ell\chi_{\ell'}\neq 0}\chi_{\ell'}u\|^2_{L^2(Q;\mathcal{H}^1)}\leq C^3_\chi\sum_{\ell'\in\mathcal{L}}\|\chi_{\ell'}u\|^2_{L^2(Q;\mathcal{H}^1)}$$

because $\sharp\left\{\ell'\in\mathcal{L}\,,\,\chi_\ell\chi_{\ell'}\neq 0\right\}$ is uniformly bounded w.r.t $\ell\in\mathcal{L}$. The right-hand side $\sum_{\ell'\in\mathcal{L}}\|\chi_{\ell'}u\|^2_{L^2(Q;\mathcal{H}^1)}$ is equivalent to $\|u\|^2_{L^2(Q;\mathcal{H}^1)}$ by Lemma 6.7. We have proved

$$\begin{aligned}&\sum_{\ell\in\mathcal{L}}\|(P_{\pm,Q,g}-z)\chi_\ell u\|^2_{L^2(Q;\mathcal{H}^{s'})}+\|\chi_\ell u\|^2_{L^2(Q;\mathcal{H}^1)}\\ &\qquad\qquad\leq C_\chi\left[(P_{\pm,Q,g}-z)u\|^2_{L^2(Q;\mathcal{H}^{s'})}+\|u\|^2_{L^2(Q;\mathcal{H}^1)}\right]<+\infty.\end{aligned}$$

b) The formula (131) applied with $P=P_{\pm,Q,g}$ and $\mathrm{ad}^2_{\chi_\ell}P_{\pm,Q,g}=|\nabla_p\chi_\ell|^2_q=-\left[\frac{1}{2^\ell}\tilde{\chi}'_{0,1}\left(\frac{|p|_q}{2^\ell}\right)\right]^2$, gives

$$P_{\pm,Q,g}-\sum_{\ell\in\mathcal{L}}\chi_\ell P_{\pm,Q,g}\chi_\ell=+\frac{1}{2}\sum_{\ell\in\mathcal{L}}\left[\frac{1}{2^\ell}\tilde{\chi}'_{0,1}\left(\frac{|p|_q}{2^\ell}\right)\right]^2\quad\in\mathrm{DiffP}^{0,-2}\,.\tag{67}$$

This yields the inequality (66).
c) Let us finish the proof of the first statement. Conversely to a) assume now $u\in L^2(Q;\mathcal{H}^1)$ and $\sum_{\ell\in\mathcal{L}}\|(P_{\pm,Q,g}-z)\chi_\ell u\|^2_{L^2(Q;\mathcal{H}^{s'})}<+\infty$. The relation

(67) implies for any $\ell'\in\mathcal{L}$,

$$\begin{aligned}\chi_{\ell'}(P_{\pm,Q,g}-z)u &= \chi_{\ell'}\sum_{\ell\in\mathcal{L}}\chi_\ell(P_{\pm,Q,g}-z)\chi_\ell u+\frac{1}{2}\left[\frac{1}{2^\ell}\tilde{\chi}_{0,1}'\left(\frac{|p|_q}{2^\ell}\right)\right]^2 u\\ &= \sum_{\chi_\ell\chi_{\ell'}\neq 0}\chi_{\ell'}\chi_\ell(P_{\pm,Q,g}-z)\chi_\ell u+\frac{1}{2}\left[\frac{1}{2^\ell}\tilde{\chi}_{0,1}'\left(\frac{|p|_q}{2^\ell}\right)\right]^2\chi_{\ell'}u\,,\end{aligned}$$

and the right-hand side is a finite sum with a number of terms uniformly bounded with respect to $\ell'\in\mathcal{L}$. Thus

$$\|\chi_{\ell'}(P_{\pm,Q,g}-z)u\|^2_{L^2(Q;\mathcal{H}^{s'})}\leq C_\chi^1\sum_{\chi_\ell\chi_{\ell'}\neq 0}\|(P_{\pm,Q,g}-z)\chi_\ell u\|^2_{L^2(Q;\mathcal{H}^{s'})}+\|\chi_{\ell'}u\|^2$$

holds for all $\ell'\in\mathcal{L}$ and summing over $\ell'\in\mathcal{L}$ gives

$$\begin{aligned}\|(P_{\pm,Q,g}-z)u\|^2_{L^2(Q;\mathcal{H}^{s'})} &\leq C_\chi^2\sum_{\ell'\in\mathcal{L}}\|\chi_{\ell'}(P_{\pm,Q,g}-z)u\|^2_{L^2(Q;\mathcal{H}^{s'})}\\ &\leq C_\chi^3\left[\sum_{\ell'\in\mathcal{L}}\|(P_{\pm,Q,g}-z)\chi_{\ell'}u\|^2_{L^2(Q;\mathcal{H}^{s'})}+\|\chi_{\ell'}u\|^2\right].\end{aligned}$$

We have proved

$$\begin{aligned}&\|(P_{\pm,Q,g}-z)u\|^2_{L^2(Q;\mathcal{H}^{s'})}+\|u\|^2_{L^2(Q;\mathcal{H}^1)}\\ &\qquad\leq C_\chi\left[\sum_{\ell\in\mathcal{L}}\|(P_{\pm,Q,g}-z)\chi_\ell u\|^2_{L^2(Q;\mathcal{H}^{s'})}+\|\chi_\ell u\|^2_{L^2(Q;\mathcal{H}^1)}\right]<+\infty\,.\end{aligned}$$

$\square$

6.4 Geometric KFP-operator on cylinders

We consider here the case when $Q=\mathbb{R}\times Q'$, with Q' compact without boundary, is endowed with the metric $g=1\oplus m(q')$, $\partial_{q^1}m\equiv 0$. The contangent bundles are respectively denoted by $X=T^*Q$ and $X'=T^*Q'$. The space $\mathcal{S}(X)$ is defined as the space of rapidly decaying (in terms of (q_1,p_1,p')) $\mathcal{C}^\infty$-functions on X. Its dual is denoted by $\mathcal{S}'(X)$.
On Q we shall use a partition of unity $\sum_{\ell\in\mathbb{Z}}\chi^2(q^1-\ell)=1$ with $\chi\in$

$\mathcal{C}_0^\infty((-\frac{1}{2},\frac{1}{2}))$.
On Q, the geometric Kramers-Fokker-Planck operator has the form

$$\begin{aligned} & P_{\pm,Q,g} = \pm p_1\partial_{q^1} + \frac{-\partial_{p_1}^2 + p_1^2 + 1}{2} + L = P_\pm^L \\ \text{with} \quad & L = K_{\pm,Q',m} = P_{\pm,Q',m} - \frac{1}{2} \\ \text{and} \quad & \mathfrak{L} = L^2(X') \quad , \quad D(L) = \left\{u \in L^2(Q',\mathcal{H}^1)\,,\, P_{\pm,Q',m}u \in L^2(X')\right\}\,. \end{aligned}$$

described in Section 4 and in Section 5. It satisfies the Hypothesis 1 because $D(L) = D(L^*) = D(P_{\pm,Q',m})$ contains $\mathcal{S}(X')$ and $\mathcal{C}_0^\infty(X')$ which are cores for the self-adjoint operator $\mathrm{Re}\,(L) = \mathcal{O}_{Q',m}$. The space $L^2(\mathbb{R},dq^1;\mathcal{H}^{1,L}) = L^2(\mathbb{R},dq^1;\mathcal{H}^{1,L^*})$ is nothing but $L^2(Q;\mathcal{H}^1)$.
The Proposition 4.7 provides the maximal accretivity and the corresponding domain of $K_{\pm,Q,g} = P_{\pm,Q,g} = K_\pm^L$ while Corollary 6.5 combined with the partition of unity $\sum_{\ell\in\mathbb{Z}}\chi^2(q^1-\ell)$ allows to check the subelliptic estimates of Hypothesis 2.

Proposition 6.10. *On the cylinder* $(Q = \mathbb{R}\times Q', g = 1\oplus m(q'))$*, the operator* $K_{\pm,Q,g} - \frac{d}{2}$ *defined by*

$$D(K_{\pm,Q,g}) = \left\{u \in L^2(Q;\mathcal{H}^1)\,,\ P_{\pm,Q,g}u \in L^2(X)\right\}$$

is maximal accretive with $K_{\pm,Q,g}^* = K_{\mp,Q,g}$.
It is the unique maximal accretive extension of $P_{\pm,Q,g}$ *defined on* $\mathcal{C}_0^\infty(X)$ *(or* $\mathcal{S}(X)$*) and it satisfies*

$$\forall u \in D(K_{\pm,Q,g})\,, \quad d\|u\|^2 \leq \|u\|_{L^2(Q;\mathcal{H}^1)}^2 = \mathrm{Re}\,\langle u\,,\,(K_{\pm,Q,g} + \frac{d}{2})u\rangle\,.$$

The estimate

$$\|(\pm\mathcal{Y}_\mathcal{E} - i\lambda)u\| + \|\mathcal{O}_{Q,g}u\| + \langle\lambda\rangle^{\frac{1}{2}}\|u\| + \|u\|_{H^{\frac{2}{3}}(Q;\mathcal{H}^0)} \leq C\|(K_{\pm,Q,g} - i\lambda)u\|$$

holds for all $u \in D(K_{\pm,Q,g})$ *and all* $\lambda \in \mathbb{R}$*. In particular,*

$$D(K_{\pm,Q,g}^*) = D(K_{\pm,Q,g}) = \left\{u \in L^2(Q;\mathcal{H}^1)\,,\, \mathcal{Y}_\mathcal{E}u\,,\, \mathcal{O}_{Q,g}u \in L^2(X)\right\}\,.$$

Remark 6.11. *This result means that Hypothesis 2 is verified with* $L_\pm = K_{\pm,Q',m}$*,* $L^2(\mathbb{R},dq;\mathcal{H}^{1,L}) = L^2(Q;\mathcal{H}^1)$ *and* $\mathcal{Q}_0 = H^{\frac{2}{3}}(Q;\mathcal{H}^0)$.

Proof. The first part, including the identity for $\mathbb{R}\mathrm{e}\,\langle u\,,\,K_{\pm,Q,g}u\rangle$, is a direct application of Proposition 4.7. Since $D(K_{\pm,\mathbb{R},1})\otimes^{alg}D(K_{\pm,Q',m})$ is dense in $D(K_{\pm,Q,g})=D(K_{\pm}^{L})$, with $L_{\pm}=K_{\pm,Q',m}$, and $\mathcal{C}_0^\infty(\mathbb{R}^2)$ (resp. $\mathcal{C}_0^\infty(X')$) is dense in $D(K_{\pm,\mathbb{R},1})$ (resp. $D(L_{\pm})=D(K_{\pm,Q',m})$), $\mathcal{C}_0^\infty(X)$ is a core for $K_{\pm,Q,g}$.
For the subelliptic estimate, we use the partition of unity $\sum_{\ell\in\mathbb{Z}}\chi_\ell^2\equiv 1$, with $\chi_\ell(q)=\chi(q^1-\ell)$. For $u\in D(K_{\pm,Q,g})$ and $\lambda\in\mathbb{R}$, Proposition 6.6 applied with $s'=0$ gives

$$\begin{aligned}C_d\|(K_{\pm,Q,g}-i\lambda)u\|^2 &\geq \|(P_{\pm,Q,g}-i\lambda)u\|^2+\|u\|^2_{L^2(Q;\mathcal{H}^1)}\\&\geq C^{-1}\sum_{\ell\in\mathbb{Z}}\|(P_{\pm,Q,g}-i\lambda)\chi_\ell u\|^2+\|\chi_\ell u\|^2_{L^2(Q;\mathcal{H}^1)}\,.\end{aligned}$$

With $\mathrm{supp}\,\chi_\ell\subset\left\{\ell-\frac{1}{2}<q^1<\ell+\frac{1}{2}\right\}$ and the q^1-translation invariance, (Q,g) can be replaced in the right-hand side by the compact manifold $(\mathbb{S}^1\times Q', g=1\oplus m)$. Thus Corollary 6.5 provides $\chi_\ell u\in D(K_{\pm,\mathbb{S}^1\times Q',g})$ and the uniform lower bound

$$\begin{aligned}C'\|(P_{\pm,\mathbb{S}^1\times Q',g}-i\lambda)\chi_\ell u\|^2\geq \|\mathcal{O}_{\mathbb{S}^1\times Q',g}\chi_\ell u\|^2+\|(\pm\mathcal{Y}_{\mathcal{E}}-i\lambda)\chi_\ell u\|^2\\+\langle\lambda\rangle\|\chi_\ell u\|^2+\|\chi_\ell u\|^2_{H^{\frac{2}{3}}(\mathbb{S}^1\times Q';\mathcal{H}^1)}\,.\end{aligned}$$

Summing over $\ell\in\mathbb{Z}$ ends the proof. $\square$

We end this section by checking Hypothesis 3 with

$$\mathcal{Q}=H^t(Q;\mathcal{H}^0))\,,\quad t\in[0,\frac{1}{9})\,.$$

The estimate (54) is a consequence of Proposition 6.10 with

$$\begin{aligned}\forall u\in D(K_{\pm,Q,g})\,,\quad \langle\lambda\rangle^{\frac{1}{4}}\|u\|_{\mathcal{Q}}\leq\langle\lambda\rangle^{\frac{1}{4}}\|u\|_{H^{\frac{1}{3}}(Q;\mathcal{H}^0)}\leq(\langle\lambda\rangle^{\frac{1}{2}}\|u\|)^{\frac{1}{2}}\|u\|^{\frac{1}{2}}_{H^{\frac{2}{3}}(Q;\mathcal{H}^0)}\\\leq C\|(K_{\pm,Q,g}-i\lambda)u\|\,.\end{aligned}$$

We still have to check the estimate (53). After identifying Q' with the submanifold $\{q^1=0\}$ of Q, the fiber bundle $\partial X=T^*_{Q'}Q$ is decomposed into

$$\partial X=T^*_{Q'}Q=T^*_{Q'}Q'\oplus_{Q'}N^*_{Q'}Q=T^*_{Q'}Q'\oplus_{Q'}(Q'\times\mathbb{R}_{p_1})\,,$$

The measures $|p_1|^\nu dq'dp=|p_1|^\nu dp_1dq'dp'$, $\nu\in\mathbb{R}$, are naturally defined on $T^*_{Q'}Q$ and the space $L^2(\mathbb{R},\frac{dp_1}{|p_1|};\mathfrak{L})$ is nothing but $L^2(\partial X,\frac{dq'dp}{|p_1|})$. Hypothesis 2 for $K_{\pm,L} = K_{\pm,Q,g}$ with $L_\pm = K_{\pm,Q',m}$ is validated by Proposition 6.10. Therefore Proposition 5.6 applies and we know that a solution to the equation (52), namely

$$(P_{\pm,Q,g} - i\lambda)v = \gamma(q', p_1, p')\delta_0(q^1) \quad \text{in } \mathcal{S}'(\mathbb{R}^2; D(L^*)') \subset \mathcal{S}'(X)$$
$$\text{with} \qquad \gamma \in L^2(\partial X, \frac{dq'dp}{|p_1|}),$$

belongs to $L^2(\mathbb{R}^2, dqdp; \mathfrak{L}) = L^2(X, dqdp)$. By adapting Theorem 1.2 of [Leb2] to the cylinder case, the above equation admits a unique solution in $\mathcal{S}'(X)$ but we do not really need this.

Proposition 6.12. *The Hypothesis 3 is satisfied by the operator $P^L_\pm = P_{\pm,Q,g}$ and the space $\mathcal{Q} = \mathcal{H}^t(Q; L^2(\mathbb{R}^{d-1}dp'))$ when $t \in [0, \frac{1}{9})$.*

Proof. The estimate (54) has already been checked.
We must prove that a solution $v \in L^2(\mathbb{R}, dq; \mathcal{H}^{1,L}) = L^2(Q; \mathcal{H}^1)$ to the equation $(P_{\pm,Q,g} - i\lambda)v = \gamma\delta_0(q^1)$ satisfies (53):

$$\|v\|_{\mathcal{Q}} \leq C_{\mathcal{Q},L}\left[\|\gamma\|_{L^2(\partial X, \frac{dq'dp}{|p_1|})} + \|v\|_{L^2(Q;\mathcal{H}^1)}\right].$$

First note

$$\|\langle p_1\rangle v\| \leq \|v\|_{L^2(Q;\mathcal{H}^1)}.$$

Set $\tilde{v} = \langle p_1\rangle^{-\frac{1}{2}} v$. It still belongs $L^2(Q; \mathcal{H}^1)$ and solves

$$(P_{\pm,Q,g} - i\lambda)\tilde{v} = \langle p_1\rangle^{-\frac{1}{2}}\gamma\delta_0(q^1) + f,$$
$$\text{with} \qquad f = \frac{p_1}{4\langle p_1\rangle^{\frac{5}{2}}}\partial_{p_1} v + \frac{1 - 3p_1^2/2}{4\langle p_1\rangle^{\frac{9}{2}}} v \in L^2(X).$$

With the cut-off function $\theta \in \mathcal{C}_0^\infty((-\frac{1}{2}, \frac{1}{2}))$, one gets

$$(P_{\pm,Q,g} - i\lambda)(\theta(q^1)\tilde{v}) = \langle p_1\rangle^{-\frac{1}{2}}\gamma\delta_0(q^1) + \theta(q^1)f + p_1\theta'(q^1)\tilde{v}, \quad (68)$$
$$(P_{\pm,Q,g} - i\lambda)((1 - \theta(q^1))\tilde{v}) = (1 - \theta(q^1))f - p_1\theta'(q^1)\tilde{v}. \quad (69)$$

The right-hand side of (69) belongs to $L^2(X, dqdp)$ while $(1-\theta(q^1))\tilde{v}$ belongs to $L^2(Q; \mathcal{H}^1)$. Hence Proposition 6.10 gives

$$\|(1 - \theta(q^1))\tilde{v}\|_{H^{\frac{2}{3}}(Q;\mathcal{H}^0)} \leq C\left[\|\tilde{v}\|_{L^2(Q;\mathcal{H}^1)} + \|f\|\right] \leq C'\|v\|_{L^2(Q;\mathcal{H}^1)}.$$

In (68), $(Q;g)$ can be replaced by the compact manifold $(\mathbb{S}^1 \times Q', g)$ and Corollary 6.5 gives

$$\begin{aligned}\|\theta(q^1)\tilde{v}\|_{H^{s+\frac{2}{3}}(Q;\mathcal{H}^0)} &\leq C_s\left[\|\langle p_1\rangle^{-\frac{1}{2}}\gamma\|_{L^2(\partial X, dq'dp)} + \|v\|_{L^2(Q;\mathcal{H}^1)}\right]\\ &\leq C_s\left[\|\gamma\|_{L^2(\partial X, \frac{dq'dp}{|p_1|})} + \|v\|_{L^2(Q;\mathcal{H}^1)}\right],\end{aligned}$$

as soon as $s < -\frac{1}{2}$ and $s \geq -\frac{2}{3}$.
We have proved for $s \in [-\frac{2}{3}, -\frac{1}{2})$

$$\begin{aligned}\|\langle p_1\rangle^{-\frac{1}{2}}v\|_{H^{s+\frac{2}{3}}(Q;\mathcal{H}^0)} &= \|\tilde{v}\|_{H^{s+\frac{2}{3}}(Q;\mathcal{H}^0)}\\ &\leq C'_s\left[\|\gamma\|_{L^2(\partial X, \frac{dq'dp}{|p_1|})} + \|v\|_{L^2(Q,\mathcal{H}^1)}\right],\end{aligned}$$

while the first estimate is $\|\langle p_1\rangle v\|_{H^0(Q;\mathcal{H}^0)} \leq \|v\|_{L^2(Q;\mathcal{H}^1)}$. The interpolation inequality

$$\|u\|_{H^{\frac{2\sigma}{3}}(Q;\mathcal{H}^0)} \leq C_\sigma \|\langle p_1\rangle^{-\frac{1}{2}}u\|_{H^\sigma(Q;\mathcal{H}^0)}^{\frac{2}{3}} \|\langle p_1\rangle u\|_{H^0(Q;\mathcal{H}^0)}^{\frac{1}{3}},$$

applied with $\sigma = s + \frac{2}{3} \in [0, \frac{1}{6})$ yields the result. $\square$

6.5 Comments

There are essentially two strategies to prove maximal, i.e. with optimal exponents, subelliptic estimates:

- a geometric approach following Hörmander in [HormIII]-Chap 27 (or Lerner in [Ler]) based on a microlocal reduction;
- a more algebraic approach after Rotschild-Stein in [RoSt] or Helffer-Nourrigat in [HeNo] based on local reduction via Taylor approximations and solving algebraic models.

In [Leb2] (and even in [Leb1] for non maximal estimates) Lebeau follows the first approach. Actually, another proof of Theorem 1.2 of [Leb2] is possible via a local point of view on T^*Q which involves a canonical transformation associated with a solution to the Hamilton-Jacobi equation $|\partial_q\varphi(q)|_q^2 = Cte$. In the case with a boundary, as it is well known in the analysis of propagation of

singularities for the wave equation (see [AnMe][Tay1][Tay2][MeSj1][MeSj2]), a solution φ shows singularities in the presence of glancing (i.e. gliding or grazing) rays. A microlocal point of view seems out of reach for general boundary value problems and we will follow the second local approach. Our strategy, is to reach some subelliptic estimates for the boundary value problem with the most flexible local approach.
Nevertheless, verifying Hypothesis 3 requires strong enough subelliptic estimates for whole space problems, in order to absorb the $-\frac{1}{2}-\varepsilon$-Sobolev singularity of $\delta_0(q^1)$. Referring to the result by Lebeau was the most efficient way. Since it was written only for compact riemannian manifolds, the most direct application including a translation invariance in the q^1-variable is the one proposed in Subsection 6.4. The local model of boundary manifolds will be $(-\infty, 0]\times\mathbb{T}^{d-1}$ rather the standard half-space $(-\infty,0]\times\mathbb{R}^{d-1}$.

7 Geometric KFP-operators on manifolds with boundary

The proof of Theorem 1.1 and Theorem 1.2 will be done in several steps relying on a careful analysis of local models for boundary manifolds.

7.1 Review of notations and outline

A neighborhood of $q_0\in\partial Q$ in the manifold with boundary $\overline{Q}=Q\sqcup\partial Q$ (resp. $\overline{X}=T^*\overline{Q}$) can be identified with a domain of $(-\infty,0]\times\mathbb{T}^{d-1}$ (resp. $(-\infty,0]\times\mathbb{T}^{2d-1}$), diffeomorphic to a domain of $(-\infty,0]\times\mathbb{R}^{d-1}$ (resp. $(-\infty,0]\times\mathbb{R}^{2d-1}$), with the corresponding global coordinates (q^1,q') (resp. (q^1,p_1,q',p')). The coordinate $q'\in\mathbb{T}^{d-1}$ means $q'\in\mathbb{R}^{d-1}$ plus periodicity conditions and we assume $q^1(q_0)=0$ and $q'(q_0)=0$.
The coordinate system (q^1,q') can be chosen so that the metric equals

$$g(q)=\begin{pmatrix}1 & 0\\ 0 & m(q^1,q')\end{pmatrix}\,, \tag{70}$$

and since it is a local description, we can assume

$$m(q^1,q')-m(0,q')\in\mathcal{C}_0^\infty((-\infty,0]\times\mathbb{T}^{d-1};\mathcal{M}_{d-1}(\mathbb{R}))\,. \tag{71}$$

The kinetic energy, the vertical (in p) harmonic oscillator operator and the Hamilton vector field on $\mathbb{R}^{2d}$ are given by

$$\begin{aligned}
&2\mathcal{E} = |p|^2_{g(q)} = |p_1|^2 + m^{ij}(q^1,q')p'_ip'_j\,,\\
&\Delta_p = \partial^2_{p_1} + \partial_{p'_i}m_{ij}(q^1,q')\partial_{p'_j} \quad , \quad \mathcal{O}_{Q,g} = \frac{-\Delta_p + |p|^2_q}{2}\,,\\
&\mathcal{Y}_{\mathcal{E}} = p_1\partial_{q^1} + m^{ij}(q^1,q')p'_i\partial_{q'^j} - \frac{1}{2}\partial_{q'^k}m^{ij}(q^1,q')p'_ip_{j'}\partial_{p'_k}\\
&\qquad\qquad -\frac{1}{2}\partial_{q^1}m^{ij}(q^1,q')p'_ip'_j\partial_{p_1}\,.
\end{aligned}$$

Since we will work with various metrics which fulfill (70)(71) with the same $m(0,q')$, the previous notation $|p|_q$ is replaced by $|p|_{g(q)}$. When $q=(0,q')\in \partial Q = Q'$, the equalities $g(0,q') = m(0,q') = g_0(q)$, $|p|_{g(q)} = |p|_{g_0(q)}$ allow to use $|p|_q$.
Following the Definition 6.2, the corresponding Kramers-Fokker-Planck operator equals

$$\begin{aligned}
P_{\pm,Q,g} &= \pm\mathcal{Y}_{\mathcal{E}} + \mathcal{O}_{Q,g}\,,\\
&= \pm p_1\partial_{q^1} + \frac{-\partial^2_{p_1} + |p_1|^2}{2} + P_{\pm,Q',m(q^1)} - \frac{1}{2}\partial_{q^1}m^{ij}(q^1,q')p'_ip'_j\partial_{p_1}\,, \quad (72)
\end{aligned}$$

with here $Q' = \mathbb{T}^{d-1}$, $Q = (-\infty,0]\times Q'$. The operator $P_{\pm,Q',m(q^1)}$ is the geometric Kramers-Fokker-Planck operator on $X' = T^*Q'$ associated with the metric $m(q^1)$ on Q'.
The phase-space $X = T^*Q$ is endowed with the metric $g\oplus g^{-1}$ and the symplectic volume $dqdp$. We shall use the notation $H^s(\overline{Q};\mathcal{H}^{s'})$, $\mathcal{S}(X)$, $\mathcal{S}'(X)$ introduced in Section 6, extended to $\mathfrak{f}$-valued functions or distributions, where $\mathfrak{f}$ is a complex Hilbert space.
When $Q' = \mathbb{T}^{d-1}$, $Q = (-\infty,0]\times Q'$ (resp. $Q = \mathbb{R}\times Q'$) and $\partial_{q^1}m\equiv 0$, we shall use the results of Section 4 and Section 5 with the identification

$$\begin{aligned}
&P_{\pm,Q,g} = P^L_{\pm} = \pm p_1\partial_{q^1} + \frac{-\partial^2_{p_1} + |p_1|^2 + 1}{2} + L_{\pm}\\
&L_{\pm} = K_{\pm,Q',m} - \frac{1}{2} \quad , \quad \mathfrak{L} = L^2(X', dq'dp';\mathfrak{f})\,,\\
&L^2(\mathbb{R}_-, dq^1;\mathcal{H}^{1,L}) = L^2(Q;\mathcal{H}^1)\\
\text{resp.}\quad &L^2(\mathbb{R}, dq^1;\mathcal{H}^{1,L}) = L^2(Q;\mathcal{H}^1)\,,
\end{aligned}$$

where $K_{\pm,Q',m} = K^*_{\mp,Q',m}$ is the maximal accretive realization of $P_{\pm,Q',m}$ provided by Proposition 6.10. We checked in Proposition 6.10 and Proposition 6.12 that Hypothesis 2 and Hypothesis 3 are true respectively with $\mathcal{Q}_0 = H^{\frac{2}{3}}(\mathbb{R}\times Q';\mathcal{H}^0)$ and $\mathcal{Q} = H^t(\mathbb{R}\times Q';\mathcal{H}^0)\,,\ t\in[0,\frac{1}{9})\,.$
The boundary ∂Q is identified with Q' and the measure $|p_1|dq'dp$ is well defined on $\partial X = T^*_{Q'}Q$. We recall that the trace of u at $q^1=0$ is denoted by γu, that j is a unitary involution on $\mathfrak{f}$. The boundary conditions are written

$$\begin{aligned}
&\gamma_{odd}u = \pm\,\mathrm{sign}\,(p_1)A\gamma_{ev}u \quad , \quad \gamma_{ev,odd}u = \Pi_{ev,odd}u\,,\\
\text{with}\quad &\Pi_{ev}\gamma(q',p_1,p') = \frac{\gamma(q',p_1,p') + j\gamma(q',-p_1,p')}{2}\,,\\
\text{and}\quad &\Pi_{odd}\gamma(q',p_1,p') = \frac{\gamma(q',p_1,p') - j\gamma(q',-p_1,p')}{2}\,;\\
\text{or}\quad &\Pi_{\mp}\gamma u = \frac{1-A}{1+A}\Pi_{\pm}\gamma u\,,\\
\text{with}\quad &\Pi_+ = \Pi_{ev} + \mathrm{sign}\,(p)\Pi_{odd} \quad , \quad \Pi_- = \Pi_{ev} - \mathrm{sign}\,(p)\Pi_{odd}\,.
\end{aligned}$$

The operator $(A, D(A))$ commutes with $\Pi_{ev,odd}$ and it is bounded and accretive in $L^2(\partial X, |p_1|dq'dp;\mathfrak{f})$. The norm of A in $\mathcal{L}(L^2(\partial X, |p_1|dq'dp;\mathfrak{f}))$ will be denoted $\|A\|$ and the framework of Hypothesis 4 contains the alternative

$$\begin{aligned}
&\textbf{either}\qquad c_A = \min\sigma(\mathbb{R}\mathrm{e}\ A) > 0\,;\\
&\textbf{or}\qquad A = 0\,.
\end{aligned}$$

The treatment of general metrics on half-cylinders, $Q = \mathbb{R}_-\times Q'$ with $Q' = \partial Q$ compact, we will assume the commutation

$$\left[A, e^{it|p|_q^2}\right] = 0 \quad ,\forall t\in\mathbb{R}\,. \tag{73}$$

By writing the momentum $p\in T^*_qQ$, $p = r\omega$ with $r = |p|_q$ and $\omega\in S^*_qQ$, the space $L^2(\partial X, |p_1|dq'dp;\mathfrak{f})$ equals

$$L^2(T^*_{\partial Q}Q, |p_1|dq'dp;\mathfrak{f}) = L^2((0,+\infty), r^{d-1}dr; L^2(S^*_{\partial Q}Q, |\omega_1|dq'd\omega;\mathfrak{f}))\,.$$

The commutation (73) combined with Hypothesis 4 means that $A = A(|p|_q)$ acts as a multiplication in the radial coordinate $r = |p|_q$ with

$$\|A(r)\|_{\mathcal{L}(L^2(S^*_{\partial Q}Q,|\omega_1|dq'd\omega;\mathfrak{f}))} \leq \|A\| \quad \text{for a.e. } r>0\,, \tag{74}$$

$$\text{with either}\quad \min\sigma(\mathbb{R}\mathrm{e}\ A(r)) \geq c_A > 0 \quad \text{for a.e. } r>0\,, \tag{75}$$

$$\text{or}\quad A(r) = 0 \quad \text{for a.e. } r>0\,. \tag{76}$$

Finally the analysis of a general manifold with boundary, will be studied with a spatial partition of unity. For this final analysis, we will assume $A = A(q, |p|_q)$ with

$$\|A(q,r)\|_{\mathcal{L}(L^2(S^*_qQ,|\omega_1|d\omega;\mathfrak{f}))} \leq \|A\| \quad \text{for a.e. } (q,r) \in \partial Q \times \mathbb{R}_+\,, \quad (77)$$

$$\text{with either } \min \sigma(\mathrm{Re}\; A(q,r)) \geq c_A > 0 \quad \text{for a.e. } (q,r) \in \partial Q \times \mathbb{R}_+\,, \quad (78)$$

$$\text{or} \qquad A(q,r) = 0 \quad \text{for a.e. } (q,r) \in \partial Q \times \mathbb{R}_+\,. \quad (79)$$

Here is the outline of the proof: We will carefully study the case of half-cylinders $Q = \mathbb{R}_- \times Q'$ with $Q' = \mathbb{T}^{d-1}$. The case when $\partial_{q^1} m \equiv 0$ will be an application of Section 5 and the result will hold under Hypothesis 4 which is more general than the one of Theorem 1.2. By assuming $A = A(|p|_q)$ with (74)(75)(76) and with a dyadic partition of unity in p, the corresponding subelliptic estimates will be written in a parameter dependent form. This allows an accurate parameter dependent analysis of some relatively bounded perturbations. In a second step, the dyadic partition of unity for a general metric m on a half-cylinder, $\mathbb{R}_- \times \mathbb{T}^{d-1}$, and a non symplectic change of variable in $X = T^*Q$ near ∂X will relate the general problem to the previous pertubative analysis. Finally the case of a general manifold will be treated by gluing the local models $Q = \mathbb{R}_- \times \mathbb{T}^{d-1}$ with a spatial partition of unity, by assuming $A = A(q, |p|_q)$ with (77)(78)(79).

7.2 Half-cylinders with $\partial_{q^1} m \equiv 0$

Let us consider the case when $Q = (-\infty, 0] \times Q'$ with $Q' = \mathbb{T}^{d-1}$ and a specific metric $g_0 = 1 \oplus^\perp m(q)$ satisfying $\partial_{q^1} m \equiv 0$.

Proposition 7.1. *Let $Q = (-\infty, 0] \times Q'$ with $Q' = \mathbb{T}^{d-1}$ be endowed with the metric $g_0 = 1 \oplus m$ with $\partial_{q^1} m \equiv 0$.*
Assume Hypothesis 4 for the operator A.
The operator $K_{\pm,A,g_0} - \frac{d}{2}$ defined by

$$D(K_{\pm,A,g_0}) = \left\{ \begin{array}{ll} u \in L^2(Q;\mathcal{H}^1)\,, & P_{\pm,Q,g_0} u \in L^2(X, dqdp; \mathfrak{f}), \\ & \gamma u \in L^2(\partial X, |p_1| dq' dp; \mathfrak{f})\,, \\ & \gamma_{odd} u = \pm\, sign\,(p_1) A \gamma_{ev} u \end{array} \right\}\,,$$

$$\forall u \in D(K_{\pm,A,g_0})\,,\; K_{\pm,A,g_0} u = P_{\pm,Q,g_0} u = (\pm \mathcal{Y}_{\mathcal{E}} + \mathcal{O}_{Q,g_0}) u\,,$$

is maximal accretive, with

$$\forall u \in D(K_{\pm,A,g_0})\,, \quad \|u\|^2_{L^2(Q;\mathcal{H}^1)} + \mathbb{R}\mathrm{e}\,\langle \gamma_{ev} u\,,\, A\gamma_{ev} u\rangle = \mathbb{R}\mathrm{e}\,\langle u\,,\, (K_{\pm,A,g_0} + \frac{d}{2})u\rangle\,.$$

The adjoint $K^*_{\pm,A,g_0}$ *equals* $K_{\mp,A^*,g_0}$ *.*
Respectively to the cases $A \neq 0$ *and* $A = 0$ *, fix* $t \in [0, \frac{1}{9})$ *and* $\nu = \frac{1}{4}$ *(resp.* $t = \frac{2}{3}$ *and* $\nu = 0$*). There exist* $C_t \geq 1$ *and* $C \geq 1$ *independent of* t *, such that*

$$\langle\lambda\rangle^{\frac{1}{4}}\|\gamma u\|_{L^2(\partial X, |p_1|dq'dp;\mathfrak{f})} + \langle\lambda\rangle^{\frac{1}{2}}\|u\| + \langle\lambda\rangle^{\frac{1}{4}}\|u\|_{L^2(Q;\mathcal{H}^1)} + C_t^{-1}\langle\lambda\rangle^{\nu}\|u\|_{H^t_q(\overline{Q};\mathcal{H}^0)} \leq C\|(K_{\pm,A,g_0} - i\lambda)u\| \quad (80)$$

for all $\lambda \in \mathbb{R}$ *and all* $u \in D(K_{\pm,A,g_0})$ *.*
For any $\Phi \in \mathcal{C}^\infty_b((-\infty, 0])$ *such that* $\Phi(0) = 0$ *there exists* $C_\Phi > 0$ *and* $C' > 0$ *independent of* Φ *, such that*

$$\|\Phi(q^1)\mathcal{O}_{Q,g_0} u\| \leq C'\|\Phi\|_{L^\infty}\|(K_{\pm,A,g_0} - i\lambda)u\| + C_\Phi\|u\|\,,$$

for all $\lambda \in \mathbb{R}$ *and all* $u \in D(K_{\pm,A,g_0})$ *.*
Finally in the case $A = 0$ *, the set* $\left\{u \in \mathcal{C}^\infty_0(\overline{X};\mathfrak{f})\,, \gamma_{odd} u = 0\right\}$ *is dense in* $D(K_{\pm,0,g_0})$ *endowed with the graph norm.*

Proof. Tensorizing with $\mathfrak{f}$ does not change the scalar results of Subsection 6.4. In Proposition 6.10 and Proposition 6.12, Hypothesis 2 and Hypothesis 3 have been checked with $L_\pm = K_{\pm,Q',m}$, $\mathcal{Q}_0 = H^{\frac{2}{3}}(\mathbb{R} \times Q';\mathcal{H}^0)$, $\mathcal{Q} = H^t(\mathbb{R} \times Q';\mathcal{H}^0)$, while Hypothesis 4 is assumed. We can refer to the results of Section 4 and Section 5. The maximal accretivity, the integration by part identity and the identification of $K_{\pm,A,g_0}$ are provided by Proposition 5.1 when $A \neq 0$ and by Proposition 4.8 when $A = 0$. The definition of the Sobolev spaces $H^s(\overline{Q};\mathcal{H}^0)$ says

$$\|u\|_{H^s(\overline{Q};\mathcal{H}^0)} \leq C_s\|\Sigma u\|_{H^s(\mathbb{R}\times Q';\mathcal{H}^0)}\,.$$

Hence the subelliptic estimate (80) and the upper bound for $\|\Phi(q^1)\mathcal{O}_{Q,g_0}\|$ are proved in Proposition 5.10 when $A \neq 0$ and in Proposition 5.5 when $A = 0$ (actually it works with $\Phi(q^1) = 1$ in this case).
Because $\mathcal{C}^\infty_0(\mathbb{R} \times Q';\mathfrak{f})$ is a core for $K_{\pm,g_0}$, Proposition 5.5 also implies that the set

$$\mathcal{D}_0 = \left\{u \in L^2(X, dqdp;\mathfrak{f})\,, \Sigma u \in \mathcal{C}^\infty_0(\mathbb{R} \times Q';\mathfrak{f})\right\}$$

is dense in $D(K_{\pm,0,g_0})$ endowed with its graph norm. This set is contained in

$$\mathcal{D}_1 = \left\{u \in \mathcal{C}^\infty_0(\overline{X};\mathfrak{f})\,, \gamma_{odd} u = 0\right\}\,,$$

and $\mathcal{D}_0 \subset \mathcal{D}_1 \subset D(K_{\pm,0,g_0})$ implies that $\mathcal{D}_1$ is dense in $D(K_{\pm,0,g_0})$. $\square$

7.3 Dyadic partition of unity and rescaled estimates

We still work on $\overline{Q} = (-\infty, 0] \times Q'$ with $Q' = \mathbb{T}^{d-1}$, with global coordinates $(q,p) \in \overline{\mathbb{R}_-} \times \mathbb{T}^{d-1} \times \mathbb{R}^d$, with a metric $g = 1 \oplus m$ which satisfies (70)(71). The notation g_0 is specific to the case $\partial_{q^1} m \equiv 0$. The operator A is assumed to satisfy Hypothesis 4 and the commutation (73) with $|p|_q^2$, which can be written as (74)(75)(76).

7.3.1 Rescaling

When $\partial_{q^1} m \equiv 0$, the maximal accretive realization $K_{\pm,A,g_0}$ is given by Proposition 7.1. The strengthened assumptions on A in (73) yields

$$(u \in D(K_{\pm,A,g_0})) \to \left(\chi(|p|^2_{g_0(q)})u \in D(K_{\pm,A,g_0})\right)$$

for all $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$. Therefore we can use a dyadic partition of unity in the momentum variable, $\chi = (\chi_\ell)_{\ell \in \mathbb{N}}$, like in Subsection 6.3.2. This will be used for the two metrics $\tilde{g} = g$ and $\tilde{g} = g_0$:

$$\begin{aligned}
&\chi_\ell(q,p) = \tilde{\chi}_1(2^{-\ell}|p|_{\tilde{g}(q)}) \text{ for } \ell \in \mathbb{N}^* \quad , \quad \chi_0(q,p) = \tilde{\chi}_0(|p|_{\tilde{g}(q)})\,,\\
&\tilde{\chi}_0 \in \mathcal{C}_0^\infty(\mathbb{R}) \quad , \quad \tilde{\chi}_1 \in \mathcal{C}_0^\infty((0,+\infty))\,,\\
&\sum_{\ell \in \mathcal{L}} \chi_\ell^2(q,p) \equiv 1 \quad , \quad \mathcal{L} = \mathbb{N}\,.
\end{aligned}$$

The equivalence of norms

$$\begin{aligned}
&\left(\frac{\sum_{\ell\in\mathcal{L}} \|\chi_\ell u\|^2_{L^2(Q;\mathcal{H}^{s'})}}{\|u\|^2_{L^2(Q;\mathcal{H}^{s'})}}\right)^{\pm 1} \leq C_{g,\chi,s'} \quad , s' \in \mathbb{R}\,,\\
&\left(\frac{\sum_{\ell\in\mathcal{L}} \|\chi_\ell u\|^2_{H^s(\overline{Q};\mathcal{H}^0)}}{\|u\|^2_{H^s(\overline{Q};\mathcal{H}^0)}}\right)^{\pm 1} \leq C_{g,\chi,s}\,, \quad s \in [-1,1]\,,
\end{aligned}$$

are given by Lemma 6.7 and Lemma 6.8.
With $\partial X = \mathbb{T}^{d-1} \times \mathbb{R}^d$, the two metrics g and g_0 coincide and the traces satisfy

$$\|\gamma\|^2_{L^2(\partial X, |p_1|dq'dp;\mathfrak{f})} = \sum_{\ell \in \mathbb{N}} \|\chi_\ell u\|^2_{L^2(\partial X, |p_1|dq'dp;\mathfrak{f})}\,.$$

By working with $\tilde{g} = g_0$, Proposition 7.1 ensures

$$\forall \lambda \in \mathbb{R}\,, \forall w \in D(K_{\pm,A,g_0})\,, \quad \|w\|_{L^2(Q;\mathcal{H}^1)} \leq C\|(K_{\pm,A,g_0} - i\lambda)w\|\,.$$

Therefore the result of Proposition 6.9 reads

$$\forall u \in D(K_{\pm,A,g_0})\,, \quad \left(\frac{\sum_{\ell\in\mathbb{R}} \|(K_{\pm,A,g_0} - i\lambda)\chi_\ell u\|^2}{\|(K_{\pm,A,g_0} - i\lambda)u\|^2}\right)^{\pm 1} \leq C_\chi\,.$$

After the unitary change of variables

$$u(q,p) = 2^{-\ell d/2} v(q, \frac{p}{2^\ell}) \quad , \quad f(q,p) = 2^{-\ell d/2}\varphi(q, \frac{p}{2^\ell})\,,$$

the equation $(K_{\pm,A,g_0} - i\lambda)u = f$, i.e. the boundary value problem

$$\begin{cases} (P_{\pm,Q,g_0} - i\lambda)u = f \\ \gamma_{ev} u = \pm\,\mathrm{sign}\,(p_1) A(|p|_q)\gamma_{ev} u\,, \end{cases}$$

becomes

$$\begin{cases} h^{-1}(P^h_{\pm,Q,g_0} - i\lambda h)v = \varphi \\ \gamma_{ev} v = \pm\,\mathrm{sign}\,(p_1) A^h(|p|_q)\gamma_{ev} v\,, \end{cases}$$

where $P^h_{\pm,Q,g_0}$, $\mathcal{O}^h_{Q,g_0}$ and A^h are defined according to the following general definition.

Definition 7.2. *For a metric $\tilde{g}$ on $(-\infty,0]\times Q'$ or $\mathbb{R}\times Q'$ with $Q' = \mathbb{T}^d$ which satisfies* (70)(71)*, a bounded operator A on $L^2(\partial X, |p_1|dq'dp; \mathfrak{f})$ and $h > 0$, we set*

$$P^h_{\pm,Q,\tilde{g}} = \pm\sqrt{h}\mathcal{Y}_{\mathcal{E}} + \mathcal{O}^h_{Q,\tilde{g}} \quad , \quad h = 2^{-2\ell}\,, \tag{81}$$

$$\mathcal{O}^h_{Q,\tilde{g}} = \frac{1}{2}\left[-h^2\Delta_p + |p|^2_{\tilde{g}(q)}\right] = \frac{1}{2}\left[-h^2\partial_p^T\tilde{g}(q)\partial_p + p^T\tilde{g}^{-1}(q)p\right] \,, \tag{82}$$

$$\textit{and} \qquad A^h(|p|_{\tilde{g}(q)}) = A(h^{-\frac{1}{2}}|p|_{\tilde{g}(q)})\,. \tag{83}$$

The operator $K^h_{\pm,A,g_0} = P^h_{\pm,Q,g_0}$, with the domain $D(K^h_{\pm,A,g_0})$ characterized by

$$u \in L^2(Q;\mathcal{H}^1)\,, \quad \begin{array}{l} P^h_{\pm,Q,g_0} u \in L^2(X, dqdp; \mathfrak{f}), \\ \gamma u \in L^2(\partial X, |p_1|dq'dp; \mathfrak{f})\,, \\ \gamma_{odd} u = \pm\,\mathrm{sign}\,(p_1) A^h \gamma_{ev} u \end{array}$$

is maximal accretive with

$$\frac{\left[\|g_0^{1/2}(q)(h\partial_p)v\|^2 + \|g_0^{-1/2}(q)pv\|^2\right]}{2} + \sqrt{h}\,\mathbb{R}\mathrm{e}\,\langle\gamma_{ev}v\,,\, A^h\gamma_{ev}v\rangle_{L^2(\partial X,|p_1|dq'dp;\mathfrak{f})}$$
$$= \mathbb{R}\mathrm{e}\,\langle u\,,\, K^h_{\pm,A,g_0}u\rangle\,.$$

Actually, $h^{-1}K^h_{\pm,A,g_0}$ is by construction unitarily equivalent to $K_{\pm,A,g_0}$, studied in Proposition 7.1.
The other estimates of Proposition 7.1 now say

$$\langle\lambda\rangle^{\frac{1}{4}}h^{3/4}\|\gamma v\|_{L^2(\partial X,|p_1|dpdq';\mathfrak{f})}+\langle\lambda\rangle^{\frac{1}{2}}h\|v\|+\langle\lambda\rangle^{\frac{1}{4}}\sqrt{h}\left[\|h\partial_p v\|+\|pv\|\right]$$
$$+C_t^{-1}\langle\lambda\rangle^{\nu}h\|u\|_{H^t(\overline{Q};\mathcal{H}^0)}\leq C\|(K^h_{\pm,A,g_0}-i\lambda h)u\| \quad (84)$$

for all $\lambda\in\mathbb{R}$ and all $u\in D(K^h_{\pm,A,g_0})$, with $(\nu,t)\in\left\{\frac{1}{4}\right\}\times[0,\frac{1}{9})$ when $A\neq 0$ and $(\nu,t)=(0,\frac{2}{3})$ when $A=0$.
Similarly for $\Phi\in\mathcal{C}_b^\infty((-\infty,0])$ such that $\Phi(0)=0$, the estimates

$$\|\Phi(q^1)\mathcal{O}^h_{Q,g_0}u\|\leq C\|\Phi\|_{L^\infty}\|(K^h_{\pm,A,g_0}-i\lambda h)u\|+C_\Phi h\|u\| \quad (85)$$

holds for all $\lambda\in\mathbb{R}$ and all $u\in D(K^h_{\pm,A,g_0})$.

7.3.2 A perturbative result for $K^h_{\pm,A,g_0}$

Remember $\overline{Q}=(-\infty,0]\times\mathbb{T}^d$ and $g_0=1\oplus m(q')$.
We shall consider now a perturbation of the operator $P^h_{\pm,Q,g_0}$:

$$\begin{aligned}P^h_{\pm,Q,g_0}(\mathcal{T}) &= \pm\sqrt{h}\left[\mathcal{Y}_{\mathcal{E}}+p_ib^i(q)\theta(|p|_{g_0(q)})+S^{ij}_k(q)\theta(|p|_{g_0(q)})p_ip_j\partial_{p_k}\right]\\ &\quad+\frac{-(h\partial_p)^T(g_0(q)+T_1(q))(h\partial_p)+p^T(g_0^{-1}(q)+T_2(q)p}{2},\\ \text{with } \mathcal{Y}_{\mathcal{E}} &= g_0^{ij}(q)p_i\partial_{q^j}-\frac{1}{2}\partial_{q^k}g_0^{ij}(q)p_ip_j\partial_{p_k}\\ &= p_1\partial_{q^1}+m^{ij}(q')p'_i\partial_{q'^j}-\frac{1}{2}\partial_{q'^k}m^{ij}(q')p'_ip'_j\partial_{p'_k},\end{aligned}$$

indexed by

$$\mathcal{T}=(b,S,\theta,T_1,T_2).$$

The collection $\mathcal{T}$ fullfills the following assumptions

$$\left\{\begin{array}{l} b\in L^\infty(Q;\mathbb{R}^d)\quad,\quad S\in L^\infty(Q;\mathbb{R}^{d^3}),\\ \theta\in\mathcal{C}_0^\infty([0,2R_\theta))\quad,\quad 0\leq\theta\leq 1\quad,\quad \theta\equiv 1 \text{ in } [0,R_\theta]\,, R_\theta\geq 1\\ T_{1,2}=T_{1,2}^T\in L^\infty(Q;\mathcal{M}_d(\mathbb{R})),\\ \|T_{1,2}\|_{L^\infty}\leq\varepsilon_T\quad,\quad |T_{1,2}(q)|\leq C_T|q^1|.\end{array}\right. \quad (86)$$

Proposition 7.3. *Let $\mathcal{T} = (b, S, \theta, T_1, T_2)$ satisfy* (86).
There exists $\varepsilon_{m,A} > 0$ fixed by the metric $g_0 = 1 \oplus^{\perp} m(q')$ and the operator A, such that the operator $K^h_{\pm,A,g_0}(\mathcal{T}) = P^h_{\pm,Q,g_0}(\mathcal{T})$, with the domain $D(K^h_{\pm,A,g_0}(\mathcal{T})) = D(K^h_{\pm,A,g_0})$, satisfies the following properties uniformly w.r.t $h \in (0,1]$ for some constant $C(\mathcal{T})$, as soon as $\varepsilon_T \leq \varepsilon_{m,A}$.
The operator $C(\mathcal{T})\sqrt{h} + K^h_{\pm,A,h}(\mathcal{T})$ is maximal accretive with

$$\begin{aligned}\mathbb{R}\mathrm{e}\,\langle v\,,\, K^h_{\pm,A,g_0}(\mathcal{T})v\rangle &\geq \sqrt{h}\,\mathbb{R}\mathrm{e}\,\langle \gamma_{ev}v\,,\, A^h\gamma_{ev}v\rangle_{L^2(\partial X,|p_1|dq'dp;\mathfrak{f})}\\ &\pm \sqrt{h}\,\mathbb{R}\mathrm{e}\,\langle v\,,\, p_i b^i(q)\theta(|p|_{g_0(q)})v\rangle \mp \frac{\sqrt{h}}{2}\langle v\,,\, S^{ij}_k(q)[\partial_{p_k}(p_ip_j\theta(|p|_{g_0(q)})]v\rangle\\ &\qquad + \frac{\|(g_0+T_1)^{1/2}(q)(h\partial_p)v\|^2 + \|(g_0^{-1}+T_2)^{1/2}(q)pv\|^2}{2}\,.\end{aligned}$$

By taking $(\nu_1,\nu_2,\nu_3,t) \in \left\{(\frac{1}{8},\frac{1}{8},\frac{3}{4})\right\} \times [0,\frac{1}{18})$ when $A \neq 0$ and $(\nu_1,\nu_2,\nu_3,t) = (0,\frac{1}{4},\frac{5}{4},\frac{1}{3})$ when $A = 0$, there exists a constant $C'_t > 0$ such that

$$\begin{aligned}N(v,\lambda,t,h) = \langle\lambda\rangle^{\frac{1}{4}}h\|v\| + \langle\lambda\rangle^{\frac{1}{8}}\sqrt{h}\,[\|h\partial_p v\| + \|pv\|] + {C'}_t^{-1}\langle\lambda\rangle^{\nu_1}h\|u\|_{H^t(\overline{Q};\mathcal{H}^0)}\\ + \langle\lambda\rangle^{\nu_2}h^{\nu_3}\|\gamma v\|_{L^2(\partial X,|p_1|dq'dp;\mathfrak{f})}\end{aligned}$$

is estimated as follows.
When $h = 1$ and $D(K^1_{\pm,A,g_0}(\mathcal{T})) = D(K_{\pm,A,g_0})$,

$$\begin{aligned}&\forall v \in D(K_{\pm,A,g_0})\,,\ \forall\lambda\in\mathbb{R}\,,\\ &\qquad N(v,\lambda,t,1) \leq C(\mathcal{T})\left[\|(K^1_{\pm,A,g_0}(\mathcal{T}) - i\lambda)v\| + \|v\|\right]\,.\end{aligned}$$

When $h \leq \frac{1}{C(\mathcal{T})}$, the operator $K^h_{\pm,A,g_0}(\mathcal{T}) + \theta(4R^2_\theta|p|_{g_0(q)}) - \frac{1}{C(\mathcal{T})}$ is maximal accretive with

$$\begin{aligned}&\forall v \in D(K_{A,h})\,,\ \forall\lambda\in\mathbb{R}\,,\\ &\qquad N(v,\lambda,t,h) \leq C(\mathcal{T})\|(K^h_{\pm,A,g_0}(\mathcal{T}) + \theta(4R^2_\theta|p|_{g_0(q)}) - i\lambda h)v\|\,.\end{aligned}$$

Proof. When $\|T_{1,2}\|_{L^\infty} \leq \varepsilon_T$ is small enough $(g_0+T_1)(q)$ and $(g_0^{-1}+T_2)(q)$ are uniformly positive. The lower bound for $\mathbb{R}\mathrm{e}\,\langle v\,,\, K^h_{\pm,A,g_0}(\mathcal{T})v\rangle$ comes from the integration by part identity for $K^h_{\pm,A,g_0}$ (the rescaled version of $K_{\pm,A,g_0}$) after

$$\begin{aligned}K^h_{\pm,A,g_0}(\mathcal{T}) - K^h_{\pm,A,g_0} = \pm\sqrt{h}p_ib^i(q)\theta(|p|_{g_0(q)}) \pm \sqrt{h}S^{ij}_k(q)p_ip_j\theta(|p|_{g_0(q)})\partial_{p_k}\\ + \frac{-(h\partial_p)^T T_1(q)(h\partial_p) + p^T T_2(q)p}{2}\,,\end{aligned}$$

where the right-hand side is a vertical operator sending $L^2(Q;\mathcal{H}^1)$ into its dual $L^2(Q;\mathcal{H}^{-1})$ with

$$[S_k^{ij}(q)p_ip_j\theta(|p|_{g_0(q)})\partial_p]^* = -S_k^{ij}(q)p_ip_j\theta(|p|_{g_0(q)})\partial_p - S_k^{ij}(q)\partial_{p_k}[p_ip_j\theta(|p|_{g_0(q)})]\,.$$

This proves the accretivity of $C_1(\mathcal{T})\sqrt{h}+K^h_{\pm,A,g_0}(\mathcal{T})$ owing to the cut-off function θ, when $C_1(\mathcal{T})$ is chosen large enough.
More precisely, this inequality also implies

$$\begin{aligned}\mathbb{R}\mathrm{e}\,\langle v\,,\,\left[\theta(4R_\theta^2|p|_{g_0(q)})+K^h_{\pm,A,g_0}(\mathcal{T})\right]v\rangle \;&\geq\; \frac{1}{2}\langle v\,,\,\left[\theta(4R_\theta^2|p|_{g_0(q)})+p^T(g_0^{-1}+T_2)pv\right]\rangle\\ &\qquad -C_1(\mathcal{T})\sqrt{h}\|v\|^2\\ &\geq\; \frac{1}{C_2(\mathcal{T})}\|v\|^2\,,\end{aligned}$$

for $C_2(\mathcal{T})$-large enough, when $h\leq h(\mathcal{T})$.
With the cut-off θ, with the estimate (85) and our assumption on T_1 and T_2, $K^h_{\pm,A,g_0}(\mathcal{T})$ is a relatively bounded perturbation of $K^h_{\pm,A,g_0}$ with relative bound $a<1$ as soon as $\varepsilon_T\leq\varepsilon_{m,A}$ and $\varepsilon_{m,A}$ is chosen small enough (the constant C in (85) is fixed by the metric $g_0=1\oplus m$ and A). Hence $C_1(\mathcal{T})\sqrt{h}+K^h_{\pm,A,g_0}(\mathcal{T})$ is a maximal accretive operator.
We still have to check the upper bound for $N(v,\lambda,t,h)$.
Case 1 ($h=1$): We already know that

$$\forall u\in D(K_{\pm,A,g_0})\,,\quad N(u,\lambda,t,1)\leq C_{m,A}\|(K_{\pm,A,g_0}-i\lambda)u\|$$

while

$$\|(K^1_{\pm,A,g_0}(\mathcal{T})-K_{\pm,A,g_0})u\|\leq C_3(\mathcal{T})\|u\|_{L^2(Q;\mathcal{H}^1)}+\|\Phi_T(q^1)\mathcal{O}_{Q,g_0}u\|$$

for some function $\Phi_T\in C_0^\infty((-\infty,0])$ such that $\Phi_T(0)=0$ and $\|\Phi_T\|_{L^\infty}\leq C_m\varepsilon_T$. Again the estimate (85) provides

$$\|\Phi_T(q^1)\mathcal{O}_{Q,g_0}u\|\leq\frac{1}{2}\|(K_{\pm,A,g_0}-i\lambda)u\|+C'_{T,m,A}\|u\|\,,$$

when $\varepsilon_T\leq\varepsilon_{m,A}$ if $\varepsilon_{m,A}$ is chosen small enough. Meanwhile the lower bound for $\mathbb{R}\mathrm{e}\,\langle u\,,\,K^1_{\pm,A,g_0}(\mathcal{T})u\rangle$ provides

$$\|u\|^2_{L^2(Q;\mathcal{H}^1)}\leq C_3(\mathcal{T})\left[\|u\|+\|(K^1_{\pm,A,g_0}(\mathcal{T})-i\lambda)u\|\right]$$

when $C_3(\mathcal{T})$ is chosen large enough. This yields

$$N(u,\lambda,t,1)\leq C_4(\mathcal{T})\left[\|u\|+\|(K^1_{\pm,A,g_0}(\mathcal{T})-i\lambda)u\|\right]\,.$$

Case 2 ($h\leq h(\mathcal{T})$): We shall distinguish the case $(b,S)=(0,0)$ from the general case, with the notations $\mathcal{T}_0=(0,0,\theta,T_1,T_2)\,,\ \mathcal{T}=(b,S,\theta,T_1,T_2)\,.$ For $C_5(\mathcal{T})$ large enough we know that $(K^h_{\pm,A,g_0}(\mathcal{T})-z)^{-1}$ and $(K^h_{\pm,A,g_0}(\mathcal{T}_0)-z)^{-1}$ are well defined for $\mathrm{Re}\ z\leq -C_5(\mathcal{T})\sqrt{h}\,.$ The proof relies on the second resolvent formula

$$\begin{aligned}(K^h_{\pm,A,g_0}(\mathcal{T}_0)-z)^{-1}&-(K^h_{\pm,A,g_0}(\mathcal{T})-z)^{-1}\\&=(K^h_{\pm,A,g_0}(\mathcal{T}_0)-z)^{-1}B(K^h_{\pm,A,g_0}(\mathcal{T})-z)^{-1}\end{aligned}\quad(87)$$

with

$$B=\pm\left[\sqrt{h}p_ib^i(q)\theta(|p|_{g_0(q)})+\frac{[S^{ij}_k(q)\theta(|p|_{g_0(q)})p_ip_j(h\partial_{p_k})]}{\sqrt{h}}\right]\,.$$

after considering the case $(b,S)=(0,0)\,.$
For $b=0$**,** $S=0$**:** Take the quantity (84),

$$\begin{aligned}N_2(v,\lambda,t,h)=\langle\lambda\rangle^{\frac12}h\|v\|+\langle\lambda\rangle^{\frac14}\sqrt{h}\,[\|h\partial_pv\|+\|pv\|]+C_{2t}^{-1}\langle\lambda\rangle^{2\nu_1}h\|u\|_{H^{2t}(Q;\mathcal{H}^1)}\\+\langle\lambda\rangle^{\frac14}h^{3/4}\|\gamma v\|_{L^2(\partial X,|p_1|dq'dp;\mathfrak{f})}\,,\end{aligned}$$

where $(2\nu_1,2t)\in\left\{\frac14\right\}\times[0,\frac19)$ when $A\neq0$ and $(2\nu_1,2t)=(0,\frac23)$ when $A=0\,.$ We know

$$\forall v\in D(K^h_{\pm,A,g_0})\,,\quad N_2(v,\lambda,t,h)\leq C\|(K^h_{\pm,A,g_0}-i\lambda h)v\|\leq C\|(K^h_{\pm,A,g_0}-z)v\|$$

for $z=-2C_5(\mathcal{T})+i\lambda\,,\ \lambda\in\mathbb{R}\,.$ Like in **Case 1**, our assumptions on T_1,T_2 imply that the difference

$$(K^h_{\pm,A,g_0}(\mathcal{T}_0)-K^h_{\pm,A,g_0})v=\frac{-(h\partial_p)^TT_1(q)(h\partial_p)+p^TT_2(q)p}{2}v$$

satisfy

$$\|(K^h_{\pm,A,g_0}(\mathcal{T}_0)-K^h_{\pm,A,g_0})v\|\leq C_g\|\Phi_T(q^1)\mathcal{O}^h_{Q,g_0}v\|$$

for some function $\Phi_T\in\mathcal{C}^\infty_0((-\infty,0])$ such that $\Phi_T(0)=0$ and $\|\Phi_T\|_{L^\infty}\leq C_m\varepsilon_T\,.$ The estimate (85) implies

$$\|(K^h_{\pm,A,g_0}(\mathcal{T}_0)-K^h_{\pm,A,g_0})v\|\leq\frac12\|(K^h_{\pm,A,g_0}-i\lambda)v\|+C_{T,m,A}\|v\|$$

if $\varepsilon_{m,A} > 0$ is chosen small enough $(\varepsilon_T \leq \varepsilon_{m,A})$. We infer

$$\begin{array}{rcl} N_2(v,\lambda,t,h) & \leq & 2C\|(K^h_{\pm,A,g_0}(\mathcal{T}_0) - i\lambda h)v\| + 2C_{T,m,A}\|v\| \\ & \leq & C_6(\mathcal{T})\|(K^h_{\pm,A,g_0}(\mathcal{T}_0) - z)v\| \end{array}$$

when $\mathbb{R}\mathrm{e}\ z \leq -2C_5(\mathcal{T})$.
L^2-estimates for general (b,S,v): The condition $\mathbb{R}\mathrm{e}\ z \leq -2C_5(\mathcal{T})$,implies

$$z \notin \sigma(K^h_{\pm,A,g_0}(\mathcal{T}_0) \cup \sigma(K^h_{\pm,A,g_0}(\mathcal{T})\,,$$

and

$$\begin{array}{rcl} \langle\lambda\rangle^{\frac{1}{4}} h^{1/2}\|v\| & \leq & (C_5(\mathcal{T}) + \langle\lambda\rangle^{\frac{1}{2}} h)\|v\| \\ & \leq & C_5(\mathcal{T})\|v\| + N_2(v,\lambda,t,h) \\ & \leq & C_7(\mathcal{T})\|(K^h_{\pm,A,g_0}(\mathcal{T}_0) - z)v\|\,, \end{array}$$

while the lower bound for $\mathbb{R}\mathrm{e}\,\langle v\,,\,K^h_{\pm,A,g_0}(\mathcal{T})v\rangle$ also gives

$$\|h\partial_p v\| + \|pv\| \leq C_7(\mathcal{T})\|(K^h_{\pm,A,g_0}(\mathcal{T}) - z)v\|\,,$$

for some $C_7(\mathcal{T})$ large enough and all $v \in D(K^h_{\pm,A,g_0})$.
Put in the second resolvent formula (87), where the worst term in B is $h^{-\frac{1}{2}}(h\partial_p)$, this implies

$$\begin{array}{rcl} \|(K^h_{\pm,A,g_0}(\mathcal{T}) - z)^{-1}\| & \leq & C_7(\mathcal{T})\|(K^h_{\pm,A,g_0}(\mathcal{T}_0) - z)^{-1}\| \times \\ & & \qquad \left[h^{-1/2}\|(h\partial_p)\circ(K^h_{\pm,A,g_0}(\mathcal{T}) - z)^{-1}\| + 1\right] \\ & \leq & \dfrac{C_8(\mathcal{T})}{2}\left[h^{-1}\langle\lambda\rangle^{-1/4} + h^{-1/2}\langle\lambda\rangle^{-1/4}\right] \\ & \leq & C_8(\mathcal{T})h^{-1}\langle\lambda\rangle^{-1/4}\,, \end{array}$$

for all z such that $\mathbb{R}\mathrm{e}\ z \leq -2C_5(\mathcal{T})$.
$H^t(\overline{Q};\mathcal{H}^0)$-estimate for general (b,S,v): The estimate

$$C_5(\mathcal{T})\|v\| + N_2(v,\lambda,t,h) \leq C_7(\mathcal{T})\|(K^h_{\pm,A,g_0}(\mathcal{T}_0) - z)v\| \quad , \quad \mathbb{R}\mathrm{e}\ z \leq -2C_5(\mathcal{T})\,,$$

implies, with the interpolation inequality $\|u\|_{H^t(\overline{Q};\mathcal{H}^0)} \leq \kappa_t\|u\|^{1/2}\|u\|^{1/2}_{H^{2t}(Q;\mathcal{H}^0)}$,

$$\begin{array}{rcl} \langle\lambda\rangle^{\nu_1} h^{1/2}\|v\|_{H^t(\overline{Q};\mathcal{H}^0)} & \leq & \kappa_t C_{2t}\left[C_5(\mathcal{T})\|v\| + C_{2t}^{-1}\langle\lambda\rangle^{2\nu_1} h\|u\|_{H^{2t}(Q;\mathcal{H}^0)}\right] \\ & \leq & \kappa_t C_{2t}\left[C_5(\mathcal{T})\|v\| + C_{2t}^{-1}\langle\lambda\rangle^{2\nu_1} h\|u\|_{H^{2t}(Q;\mathcal{H}^0)}\right] \\ & \leq & \kappa_t C_{2t} C_7(\mathcal{T})\|(K^h_{\pm,A,g_0}(\mathcal{T}_0) - z)v\|\,. \end{array}$$

Applying again the second resolvent formula (87) leads to

$$C'_t\|(K^h_{\pm,A,g_0}(\mathcal{T})-z)^{-1}f\|_{H^t(\overline{Q};\mathcal{H}^0)}\leq C_9(\mathcal{T})\langle\lambda\rangle^{-\nu_1}h^{-1}\|f\|\,,$$

for some new function $t\in[0,\frac{1}{18})\to C'_t$, when $\mathbb{R}\mathrm{e}\, z\leq -2C_5(\mathcal{T})$ and $f\in L^2(X,dqdp;\mathfrak{f})$.

Rough trace estimates for general (b,S,v)**:** The following argument holds for both cases $A=0$ and $A\neq 0$. The inequality

$$\langle\lambda\rangle^{\frac{1}{4}}h^{\frac{3}{4}}\|\gamma_{ev}v\|_{L^2(\partial X,|p_1|dq'dp;\mathfrak{f})}\leq N_2(v,\lambda,t,h)\leq C_7(\mathcal{T})\|(K^h_{\pm,A,g_0}(\mathcal{T}_0-z)v\|$$

for $v\in D(K^h_{\pm,A,g_0})$ and $\mathbb{R}\mathrm{e}\, z\leq -2C_5(\mathcal{T})$ can be written

$$\|\gamma(K^h_{\pm,A,g_0}(\mathcal{T}_0)-z)^{-1}f\|_{L^2(\partial X,|p_1|dq'dp;\mathfrak{f})}\leq C_7(\mathcal{T})\langle\lambda\rangle^{-\frac{1}{4}}h^{-\frac{3}{4}}\|f\|\,,$$

with $f\in L^2(X,dqdp;\mathfrak{f})$. Inserted in the second resolvent formula (87), this implies

$$\|\gamma(K^h_{\pm,A,g_0}(\mathcal{T})-z)^{-1}f\|_{L^2(\partial X,|p_1|dq'dp;\mathfrak{f})}\leq C_{10}(\mathcal{T})\langle\lambda\rangle^{-\frac{1}{4}}h^{-\frac{5}{4}}\|f\|\,,$$

when $\mathbb{R}\mathrm{e}\, z\leq -2C_5(\mathcal{T})$. Especially in the case $A=0$, this proves the upper bound of $\langle\lambda\rangle^{\nu_2}h^{\nu_3}\|\gamma v\|_{L^2(\partial X,|p_1|dq'dp;\mathfrak{f})}$ with $(\nu_2,\nu_3)=(\frac{1}{4},\frac{5}{4})$ when $v\in D(K^h_{\pm,A,g_0}(\mathcal{T}))=D(K^h_{\pm,A,g_0})$.

Using the integration by part inequality for general (b,S,v)**:** For $\mathbb{R}\mathrm{e}\, z\leq -2C_5(\mathcal{T})$, the lower bound for $\mathbb{R}\mathrm{e}\,\langle v\,,\,K^h_{\pm,A,g_0}(\mathcal{T})v\rangle$ leads to

$$\begin{aligned}\|v\|\|(K^h_{\pm,A,g_0}(\mathcal{T})-z)v\| &\geq \mathbb{R}\mathrm{e}\,\langle v\,,\,(K^h_{\pm,A,g_0}(\mathcal{T})+2C_5(\mathcal{T}))v\rangle\\ &\geq \sqrt{h}\,\mathbb{R}\mathrm{e}\,\langle\gamma_{ev}v\,,\,A^h\gamma_{ev}v\rangle_{L^2(\partial X,|p_1|dq'dp);\mathfrak{f}}\\ &\quad+\frac{1}{C_{11}(\mathcal{T})}\left[\|h\partial_p v\|^2+\|pv\|^2\right]\,.\end{aligned}$$

From

$$\|v\|\leq C_8(\mathcal{T})\langle\lambda\rangle^{-\frac{1}{4}}h^{-1}\|(K^h_{\pm,A,g_0}(\mathcal{T})-z)v\|\,,$$

we deduce

$$\|h\partial_p v\|+\|pv\|\leq\sqrt{2C_{11}(\mathcal{T})C_8(\mathcal{T})}\langle\lambda\rangle^{-\frac{1}{8}}h^{-\frac{1}{2}}\|(K^h_{\pm,A,g_0}(\mathcal{T})-z)v\|$$

for all $v\in D(K^h_{\pm,A,g_0}(\mathcal{T}))=D(K^h_{\pm,A,g_0})$.

When $A\neq 0$, this improves the estimate for $\|\gamma v\|_{L^2(\partial X,|p_1|dq'dp;\mathfrak{f})}$ with

$$\begin{aligned}\frac{c_A}{2(1+\|A\|^2)}\|\gamma v\|^2_{L^2(\partial X,|p_1|dq'dp;\mathfrak{f})} &\leq \langle\gamma_{ev}v\,,\,A^h\gamma_{ev}v\rangle\\ &\leq C_8(\mathcal{T})\langle\lambda\rangle^{-\frac{1}{4}}h^{-\frac{3}{2}}\|(K^h_{\pm,A,g_0}(\mathcal{T})-z)v\|^2\,,\end{aligned}$$

and we can take the pair of exponents $(\nu_2, \nu_3) = (\frac{1}{8}, \frac{3}{4})$.
Summary: We have proved

$$N(v, \lambda, t, h) \leq C_{12}(\mathcal{T}) \|(K^h_{\pm,A,g_0}(\mathcal{T}) - z)v\|$$

when $v \in D(K^h_{\pm,A,g_0})$ and $\mathbb{R}\mathrm{e}\ z \leq -2C_5(\mathcal{T})$ and $C_{12}(\mathcal{T})$ is large enough. Take for $C_{12}(\mathcal{T})$, the maximum value of all the intermediate $C_k(\mathcal{T})$ and $\frac{2(1+\|A\|^2)}{c_A}$ when $A \neq 0$.
Taking $z = i\lambda$ for $K^h_{\pm,A,g_0}(\mathcal{T}) + \theta(4R^2_\theta |p|_{g_0(q)})$: The operator $\theta(4R^2_\theta |p|_{g_0(q)})$ is a bounded perturbation of $K^h_{\pm,A,g_0}(\mathcal{T})$ such that

$$\mathbb{R}\mathrm{e}\, \langle v\,,\, \left[K^h_{\pm,A,g_0}(\mathcal{T}) + \theta(4R^2_\theta |p|_{g_0(q)})\right] v\rangle \geq \frac{1}{C_2(\mathcal{T})} \|v\|^2\,,$$

when $h \leq h(\mathcal{T})$. This implies

$$\|(K^h_{\pm,A,g_0}(\mathcal{T}) + \theta(4R^2_\theta |p|_{g_0(q)}) - i\lambda)v\| \geq \frac{1}{C_2(\mathcal{T})} \|v\|$$

Therefore there exists $C_{13}(\mathcal{T})$ such that

$$\begin{aligned} C_{13}(\mathcal{T}) &\|(K^h_{\pm,A,g_0}(\mathcal{T}) + \theta(4R^2_\theta |p|_{g_0(q)}) - i\lambda)v\| \\ &\geq C_{12}(\mathcal{T}) \|(K^h_{\pm,A,g_0}(\mathcal{T}) + 2C_5(\mathcal{T}) - i\lambda)v\| \geq N(v, \lambda, t, h)\,. \end{aligned}$$

We finally take

$$C(\mathcal{T}) = \max\left\{\frac{1}{h(\mathcal{T})}\,, C_{12}(\mathcal{T})\,, C_{13}(\mathcal{T})\right\}\,,$$

where $C_{12}(\mathcal{T})$ depends on A in the case $A \neq 0$. $\square$

7.4 General local metric on half-cylinders

We prove the maximal accretivity and subelliptic estimates for Kramers-Fokker-Planck operators on $\overline{Q} = (-\infty, 0] \times Q'$, $Q' = \mathbb{T}^{d-1}$, endowed with metric $g = 1 \oplus m(q^1, q')$ which fulfill (70)(71) without the condition $\partial_{q^1} m \equiv 0$. Actually we shall work locally near $Q' = \{q^1 = 0\}$ and introduce a parameter $\varepsilon > 0$ to be fixed within the proof. The partial metric $m(q^1, q')$ can be written

$$\begin{aligned} & m(q^1, q') = m_0(q') + q^1 \tilde{m}(q^1, q')\,, \\ \text{with} \quad & \tilde{m} \in \mathcal{C}^\infty_0(\overline{Q}; \mathcal{M}_{d-1}(\mathbb{R}))\,. \end{aligned}$$

Since we are interested in the problem near the boundary, we can replace $m(q^1,q')$ by

$$m_\varepsilon(q^1,q')=m_0(q')+\chi_m(\frac{q^1}{\varepsilon})q^1\tilde{m}(q^1,q')\,,$$

with $\chi_m\in\mathcal{C}_0^\infty((-1,0])$, $\chi_m\equiv 1$ in a neighborhood of $\operatorname{supp}\tilde{m}$, so that $g_1=g$ while $g_0=1\oplus m_0(q')$ satisfies $\partial_{q^1}m_0\equiv 0$. The metrics m and m_ε (resp. g and g_ε) coincide in $\{|q^1|\leq C_g\varepsilon\}$. In all our estimates, the constants determined by the metric g and independent of $\varepsilon\in[0,1]$ will be denoted with a subscript $_g$ while the dependence with respect to ε for the metric g_ε will be traced carefully.

The usefull properties of the metric g_ε for $\varepsilon\in[0,1]$ are summarized by

$$\begin{aligned}
&g_\varepsilon=\begin{pmatrix}1&0\\0&m_\varepsilon(q^1,q')\end{pmatrix}\,,\quad m_\varepsilon(0,q')=m_0(q')\,,\\
&m_\varepsilon(q^1,q')-m_\varepsilon(0,q')\in\mathcal{C}_0^\infty(\overline{Q};\mathcal{M}_{d-1}(\mathbb{R}))\,,\\
&\operatorname{supp}\left[m_\varepsilon(q^1,q')-m_\varepsilon(0,q')\right]\subset\left\{|q^1|\leq C_g\varepsilon\right\}\\
&|\partial_q m_\varepsilon(q)|\leq C_g\quad,\quad |m_\varepsilon(q^1,q')-m_\varepsilon(0,q')|\leq C_g|q^1|\leq C_g\varepsilon\,.
\end{aligned}$$

The spaces $H^s(\overline{Q};\mathcal{H}^{s'})$ do not depend on g_ε but their norms $\|u\|_{H^s(\overline{Q};\mathcal{H}^{s'}),g_\varepsilon}$ do. The above estimates ensures the uniform equivalence of norms

$$\left(\frac{\|u\|_{H^s(\overline{Q};\mathcal{H}^{s'}),g_\varepsilon}}{\|u\|_{H^s(\overline{Q};\mathcal{H}^{s'}),g_0}}\right)^{\pm 1}\leq C_{s,s'}\quad\text{when } (s,s')\in[-1,1]\times\mathbb{R}\,,$$

and since only the case $s\in[-1,1]$ occurs, we keep the notation $\|u\|_{H^s(\overline{Q};\mathcal{H}^{s'})}$ without specifying the metric. The choice of the norm $\|u\|_{L^2(Q;\mathcal{H}^1)}$ in the integration by parts inequality (or identity) will be clear from the context. With $g_\varepsilon(0,q')=m_0(q')$, the scalar product $\langle\gamma\,,\,\gamma'\rangle_{L^2(\partial X,|p_1|dq'dp;\mathfrak{f})}$ does not depend on ε.

The operator A involved in the boundary condition satisfies Hypothesis 4 and (73) and is denoted $A(|p|_q)$. The notations A^h, $\mathcal{O}^h_{Q,g_\varepsilon}$ and $P^h_{\pm,Q,g_\varepsilon}$ are the ones of Definition 7.2 with $A=A^1$, $\mathcal{O}^1_{Q,g_\varepsilon}=\mathcal{O}_{Q,g_\varepsilon}$ and $P^1_{\pm,Q,g_\varepsilon}=P_{\pm,Q,g_\varepsilon}$. The result of this section is

Proposition 7.4. *Assume that A fulfills Hypothesis 4 and* (73)*. There exists $\varepsilon_g>0$ such that for $\varepsilon\leq\varepsilon_g$ the conclusions of Theorem 1.1 and Theorem 1.2 are true when $\overline{Q}=(-\infty,0]\times\mathbb{T}^{d-1}$ is endowed with the metric g_ε.*

This will be done in several steps: We first prove that for $\varepsilon \leq \varepsilon_g$ small enough, $K_{\pm,A,g_\varepsilon}$ is maximal accretive with most of the subelliptic estimates of Theorem 1.1 and Theorem 1.2. Then we prove the estimate of $\|\Phi(q)\mathcal{O}_{Q,g_\varepsilon}u\|$, the equality $K^*_{\pm,A,g_\varepsilon} = K_{\mp,A^*,g_\varepsilon}$ and the density of $\mathcal{D}(\overline{X},j)$ in $D(K_{\pm,0,g_\varepsilon})$.

7.4.1 Maximal accretivity and first subelliptic estimates

Assuming Hypothesis 4 and (73) for A, the domain of $K_{\pm,A,g_\varepsilon}$ is now defined without prescribing a global estimate for γu:

$$D(K_{\pm,A,g_\varepsilon}) = \left\{ \begin{array}{ll} u \in L^2(Q;\mathcal{H}^1)\,, & P_{\pm,Q,g}u \in L^2(X,dqdp;\mathfrak{f})\,, \\ & \gamma u \in L^2_{loc}(\partial X, |p_1|dq'dp;\mathfrak{f})\,, \\ & \gamma_{odd}u = \pm\,\mathrm{sign}\,(p_1)A\gamma_{ev}u\,. \end{array} \right\} \tag{88}$$

Proposition 7.5. *Assume that A fulfills Hypothesis 4 and* (73) *and that $\overline{Q} = (-\infty,0]\times\mathbb{T}^{d-1}$ is endowed with the metric g_ε.*
There exists $\varepsilon_g > 0$ such that the following properties hold when $\varepsilon \leq \varepsilon_g$.
The operator $K_{\pm,A,g_\varepsilon} - \frac{d}{2}$, with $K_{\pm,A,g_\varepsilon}u = P_{\pm,Q,g_\varepsilon}u$ and the domain $D(K_{\pm,A,g_\varepsilon})$ given according to (88) *is maximal accretive.*
When $\mathrm{supp}\,u \subset \{g^{ij}_\varepsilon(q)p_ip_j \leq R_u^2\}$ for some $R_u \in (0,+\infty)$, $u \in D(K_{\pm,A,g_\varepsilon})$ is equivalent to $u \in D(K_{\pm,A,g_0})$. The set of such u's is dense in $D(K_{\pm,A,g_\varepsilon})$ endowed with its graph norm.
The identity

$$\|u\|^2_{L^2(Q;\mathcal{H}^1)} + \mathbb{R}\mathrm{e}\,\langle \gamma_{ev}u\,,\, A\gamma_{ev}u\rangle_{L^2(\partial X,|p_1|dq'dp;\mathfrak{f})} = \mathbb{R}\mathrm{e}\,\langle u\,,\,(\frac{d}{2} + K_{\pm,A,g_\varepsilon})u\rangle\,, \tag{89}$$

where the left-hand side is larger thant $d\|u\|^2$, holds for all $u \in D(K_{\pm,A,g_\varepsilon})$.
The adjoint of $K^*_{\pm,A,g_\varepsilon}$ equals $K_{\mp,A^*,g_\varepsilon}$.
By taking $(\nu_1,\nu_2,\tilde{\nu}_3,t) \in \left\{(\frac{1}{8},\frac{1}{8},0)\right\}\times[0,\frac{1}{18})$ when $A \neq 0$, and $(\nu_1,\nu_2,\tilde{\nu}_3,t) = (0,\frac{1}{4},-1,\frac{1}{3})$ when $A = 0$, there exist a constant $C_t > 0$ and a constant $C > 0$ independent of t, such that the quantity

$$\begin{aligned} N_{t,g_\varepsilon}(u,\lambda) = \langle\lambda\rangle^{\frac{1}{4}}\|u\| + \langle\lambda\rangle^{\frac{1}{8}}\|u\|_{L^2(Q;\mathcal{H}^1)} + C_t^{-1}\langle\lambda\rangle^{\nu_1}\|u\|_{H^t(\overline{Q};\mathcal{H}^0)} \\ + \langle\lambda\rangle^{\nu_2}\|(1+|p|_q)^{\tilde{\nu}_3}\gamma u\|_{L^2(\partial X,|p_1|dq'dp;\mathfrak{f})} \end{aligned} \tag{90}$$

is less than $C\|(K_{\pm,A,g_\varepsilon} - i\lambda)u\|$, for all $u \in D(K_{\pm,A,g_\varepsilon})$ and all $\lambda \in \mathbb{R}$.

When $\mathcal{T} = (b,S,T_1,T_2,\theta)$ fulfills the conditions (86), $P^h_{\pm,Q,g_0}(\mathcal{T})$ (resp. $K^h_{\pm,A,g_0}(\mathcal{T})$) denotes the perturbation of $P^h_{\pm,Q,g_0}$ (resp. $K^h_{\pm,A,g_0}$) studied in Subsection 7.3.2.

Lemma 7.6. *Consider the ball*

$$B_R = \left\{ g_\varepsilon^{ij}(q) p_i p_j \leq R^2 \right\} \quad , \quad R \in (0, +\infty)\,,$$

and the unitary change of variable

$$(U_{g_\varepsilon} v)(q,p) = \det(\Psi(q))^{-1/2} v(q, \Psi(q)^{-1} p)\,, \tag{91}$$

$$\Psi(q) = g_\varepsilon(q) g_0(q)^{-1}\,. \tag{92}$$

There exists $\mathcal{T} = (b, S, T_1, T_2, \theta)$ fulfilling the conditions (86)*, with $R_\theta = C_g R$ and $\varepsilon_T \leq C_g \varepsilon$, such that*

$$K_{\pm,A,g_\varepsilon} u = U_{g_\varepsilon} K^1_{\pm,A,g_0}(\mathcal{T}) U^*_{g_\varepsilon} u\,,$$

for all $u \in D(K_{\pm,A,g_\varepsilon})$ with $\operatorname{supp} u \subset B_R$.
There exists a constant C_{R,g_ε} independent of (λ, t, u) such that the quantity (90) *is estimated by*

$$N_{t,g_\varepsilon}(u, \lambda) \leq C_{R,g_\varepsilon} \left[\|(K_{\pm,A,g_\varepsilon} - i\lambda) u\| + \|u\| \right]\,.$$

for all $\lambda \in \mathbb{R}$ and all $u \in D(K_{\pm,A,g_\varepsilon})$ such that $\operatorname{supp} u \subset B_R$. Moreover the integration by parts identity (89) *holds for such u's.*

Proof. The unitary change of variable

$$\begin{aligned}
u(q,p) &= (U_{g_\varepsilon} v)(q,p) = \det(\Psi(q))^{-1/2} v(q, \Psi(q)^{-1} p) \\
&= \det(\mu(q))^{-1/2} v(q, p_1, \mu(q)^{-1} p')\,, \\
\Psi(q) &= g_\varepsilon(q) g_0(q)^{-1} = g_\varepsilon(q^1, q') g(0, q')^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \mu(q) \end{pmatrix} \\
\mu(q) &= m_\varepsilon(q) m_0(q)^{-1} = m_\varepsilon(q^1, q') m_0(q')^{-1}\,,
\end{aligned}$$

does not change the functional spaces $L^2(Q; \mathcal{H}^{s'})$, $s' \in \mathbb{R}$, $L^2(\partial X, |p_1| dq' dp)$, nor the traces of $\gamma_{ev} u$ and $\gamma_{odd} u$ because $\mu(q) = \mathrm{Id}_{\mathbb{R}^{d-1}}$ when $q^1 = 0$. The norms of U_{g_ε} and $U_{g_\varepsilon}^{-1}$ as a bounded operator in $L^2(Q; \mathcal{H}^{s'})$ are uniformly bounded by $C_{s',g} \geq 1$ when $\varepsilon \in (0, \varepsilon_g]$. The support condition $\operatorname{supp} u \subset \{ g_\varepsilon^{ij}(q) p_i p_j \leq R^2 \}$ tranformed into $\operatorname{supp} v \subset \left\{ g_0^{ij}(q) p_i p_j \leq R_\theta^2 \right\}$ for $R_\theta \leq C_g R$. A direct calculation gives

$$\begin{aligned}
U_{g_\varepsilon}^{-1} P_{\pm,Q,g_\varepsilon} U_{g_\varepsilon} &= P_{\pm,Q,g_0} \pm p_i b^i(q) \pm S_k^{ij}(q) p_i p_j \partial_{p_k} \\
&\qquad + \frac{-(\partial_p)^T T_1(q) (\partial_p) + p^T T_2(q) p}{2}\,, \\
\text{with} \qquad & T_1(q) = g_0(q) [g_\varepsilon(q)^{-1} - g_0(q)^{-1}] g_0(q)\,, \\
& T_2(q) = g_0(q)^{-1} [g_\varepsilon(q) - g_0(q)] g_0(q)^{-1}\,, \\
& \|b\|_{L^\infty} + \|S\|_{L^\infty} \leq C_g\,.
\end{aligned}$$

The support condition on u (and v) allows to replace $p_ib^i(q)$ by $\theta(|p|_{g_0(q)})p_ib^i(q)$ and $S^{ij}(q)p_ip_j\partial_{p_k}$ by $S^{ij}(q)\theta(|p|_{g_0(q)})p_ip_j\partial_{p_k}$.
We also deduce

$$|T_1(q)| + |T_2(q)| \leq C_g|q^1| \leq C_g\varepsilon\,.$$

Since $\Psi(q) = \mathrm{Id}_{\mathbb{R}^d}$ along $\{q^1 = 0\}$, the traces are not changed and the boundary condition for $u \in D(K_{\pm,A,g_\varepsilon})$ becomes $v \in D(K_{\pm,A,g_0}) = D(K^1_{\pm,A,g_0}(\mathcal{T}))$. We are exactly in the case of Proposition 7.3 with $h = 1$, as soon as $\varepsilon \leq \varepsilon_g$ for ε_g small enough. With

$$U_g^{-1}K_{\pm,A,g}U_{g_\varepsilon}v = K^1_{\pm,A,g_0}(\mathcal{T})v\,,$$

the estimate of $N_{t,g_\varepsilon}(u,\lambda) \leq C_{R,g_\varepsilon}N(v,\lambda,t,h=1)$ is given by Proposition 7.3. For the identity (89), assume $\mathrm{supp}\,u \subset B_R$ and compute

$$\begin{aligned}
&\mathrm{Re}\,\langle u\,,\,[\frac{d}{2} + K_{\pm,A,g}]u\rangle - \|u\|^2_{L^2(Q;\mathcal{H}^1)} - \mathrm{Re}\,\langle\gamma_{ev}u\,,\,A\gamma_{ev}u\rangle_{L^2(\partial X,|p_1|dq'dp;\mathfrak{f})}\\
&= \mathrm{Re}\,\langle v\,,\,[\frac{d}{2} + K^1_{\pm,A,g_0}(\mathcal{T})]v\rangle - \|u\|^2_{L^2(\mathbb{R}^d_-;\mathcal{H}^1)}\\
&\qquad\qquad\qquad\qquad - \mathrm{Re}\,\langle\gamma_{ev}v\,,\,A\gamma_{ev}v\rangle_{L^2(\partial X,|p_1|dq'dp;\mathfrak{f})}\\
&= \pm\langle v\,,\,p_ib^i(q)v\rangle \mp \frac{1}{2}\langle v\,,\,(S^{kj}_k(q)p_j + S^{ik}_k(q)p_i)v\rangle\,.
\end{aligned}$$

The same computation done for $u \in \mathcal{C}^\infty_0(X;\mathfrak{f})$ leads to a vanishing right-hand side. Since this right-hand side is continuous on $L^2(Q;\mathcal{H}^1)$, it always vanishes when $u \in D(K_{\pm,A,g_\varepsilon}) \subset L^2(Q;\mathcal{H}^1)$ and $\mathrm{supp}\,u \subset B_R$. ☐

Lemma 7.7. *Consider the shell*

$$S_{R,\ell} = \left\{R^{-2}2^{2\ell} \leq g^{ij}_\varepsilon(q)p_ip_j \leq R^2 2^{2\ell}\right\}\,,\quad R \in (0,+\infty)\,,\quad \ell \in \mathbb{N}\,,$$

and the unitary change of variable $U_{g_\varepsilon,\ell} = U_{g_\varepsilon}V_\ell$ where U_{g_ε} is defined by (91)(92) *and $V_\ell w = 2^{-\ell d/2}w(q,2^{-\ell}p)$.*
There exists $\mathcal{T} = (b,S,T_1,T_2,\theta)$ fulfilling the conditions (86)*, with $R_\theta = C_gR$ and $\varepsilon_T \leq C_g\varepsilon$, such that*

$$K_{\pm,A,g_\varepsilon}u = \frac{1}{h}U_{g_\varepsilon,\ell}\left[\theta(4R_\theta^2|p|_{g_0(q)}) + K^h_{\pm,A,g_0}(\mathcal{T})\right]U^*_{g_\varepsilon,\ell}u\quad,\quad h = 2^{-2\ell}\,.$$

for all $u \in D(K_{\pm,A,g_\varepsilon})$ with $\mathrm{supp}\,u \subset S_{R,\ell}$.
There exists two constants C_{R,g_ε} and ℓ_{R,g_ε} , independent of (λ,t,u,ℓ) , such that the quantity (90) *is estimated by*

$$N_{t,g_\varepsilon}(u,\lambda) \leq C_{R,g_\varepsilon}\|(K_{\pm,A,g_\varepsilon} - i\lambda)u\|\,,$$

for all $\lambda \in \mathbb{R}$ for all $u \in D(K_{\pm,A,g_\varepsilon})$ such that $\operatorname{supp} u \subset S_{R,\ell}$ as soon as $\ell \geq \ell_{R,g_\varepsilon}$.
Moreover the operator $\theta(4R_\theta^2|p|_{g_0(q)}) + K^h_{\pm,A,g_0}(\mathcal{T})$, with $h = 2^{-2\ell}$, is maximal accretive when $\ell \geq \ell_{R,g_\varepsilon}$.

Proof. We start with the same unitary change of variable U_{g_ε} as in Lemma 7.6. With $u = U_{g_\varepsilon} v$ one gets

$$U_{g_\varepsilon}^{-1} K_{\pm,A,g_\varepsilon} U_{g_\varepsilon} v = K^1_{\pm,A,g_0}(\mathcal{T}_\ell) v$$

where $\mathcal{T}_\ell$ equals $(b, S, T_1, T_2, \theta(2^{-\ell}.))$ while $\mathcal{T} = (b, S, T_1, T_2, \theta)$ satisfies the same conditions as in Lemma 7.6. Additionally one can choose $\varepsilon_g > 0$ small enough and $R_\theta \leq C_g R$ such that $\operatorname{supp} u \subset S_{R,\ell}$ implies

$$\operatorname{supp} v \subset \left\{R_\theta^{-2} 2^{2\ell} \leq g_0^{ij}(q) p_i p_j \leq R_\theta^2 2^{2\ell}\right\} .$$

The function v is supported in $\left\{1 \leq R_\theta 2^{-\ell}|p|_{g_0(q)} \leq R_\theta^2\right\}$ while $\theta(4R_\theta^2 2^{-\ell}|p|_{g_0(q)})$ is supported in $\left\{R_\theta 2^\ell |p|_{g_0(q)} \leq \frac{1}{2}\right\}$. We obtain

$$U_{g_\varepsilon}^{-1} K_{\pm,A,g_\varepsilon} U_{g_\varepsilon} v = \left[\theta(4R_\theta^2 2^{-\ell}|p|_{g_0(q)}) + K^1_{\pm,A,g_0}(\mathcal{T}_\ell)\right] v$$

The unitary change of variable $v = V_\ell w = 2^{-\ell d/2} w(q, 2^{-\ell} p)$ gives

$$V_\ell^{-1} U_{g_\varepsilon}^{-1} K_{\pm,A,g_\varepsilon} U_{g_\varepsilon} V_\ell w = \frac{1}{h}\left[\theta(4R_\theta^2|p|_{g_0(q)}) + K^h_{\pm,A,g_0}(\mathcal{T})\right] w$$

with $h = 2^{-2\ell}$. After noticing that $N_{t,g_\varepsilon}(u,\lambda) \leq \frac{C_{g_\varepsilon,R} N(v,\lambda,t,h)}{h}$ with $h = 2^{-2\ell}$, it suffices to take $U_{g_\varepsilon,\ell} = U_{g_\varepsilon} V_\ell$ and to apply the subelliptic estimates of Proposition 7.3 for $K^h_{\pm,A,g_0}(\mathcal{T}) - i\lambda h$ which are valid for $h \leq \frac{1}{C(\mathcal{T})}$ small enough.
The maximal accretivity of $\left[\theta(4R_\theta^2|p|_{g_0(q)}) + K^h_{\pm,A,g_0}(\mathcal{T})\right]$ was also checked in Proposition 7.3 for $h \leq \frac{1}{C(\mathcal{T})}$, which means simply $\ell \geq \ell_{R,g_\varepsilon}$. $\square$

Proof of Proposition 7.5:
We fix $\varepsilon \leq \varepsilon_g$ with ε_g small enough so that Lemma 7.6 and Lemma 7.7 apply. The cylinder $\overline{Q} = (-\infty,] \times \mathbb{T}^d$ is then endowed with the metric g_ε. The quantities $|p|_{g_\varepsilon(q)}$ and $|p|_{g_0(q)}$ satisfy uniformly with respect to $(q,\varepsilon) \in Q' \times [0, \varepsilon_g]$, $\left(\frac{|p|_{g_\varepsilon(q)}}{|p|_{g_0(q)}}\right)^{\pm 1} \leq (1 + C_g \varepsilon)$.
Dyadic partition of unity: The dyadic partition of unity $\chi = (\chi_\ell)_{\ell\in\mathbb{N}}$ with $\mathbb{N}$ is given like in Paragraph 6.3.2 by $\chi_0(q,p) = \tilde{\chi}_0(|p|_{g_\varepsilon(q)})$ and $\chi_\ell(q,p) =$

$\tilde{\chi}_1(2^{-\ell}|p|_{g_\varepsilon(q)})$ with $\sum_{\ell\in\mathbb{N}}\chi_\ell^2 \equiv 1$. The cut-off functions $\tilde{\chi}_0$ and $\tilde{\chi}_1$ are assumed to be supported respectively in $\left\{|p|_{g_\varepsilon(q)} \leq R_0\right\}$ and $\left\{R_1^{-1} \leq |p|_{g_\varepsilon(q)} \leq R_1\right\}$ for some $R_1 \geq 1$.
By applying Lemma 7.7 with $R = 2R_1$, there exists $\ell_{\chi,g_\varepsilon} = \ell_{2R_1,g_\varepsilon} \geq 1$ and $\mathcal{T}_1 = (b, S, T_1, T_2, \theta)$ such that for $\ell \geq \ell_{\chi,g_\varepsilon}$ the operator

$$K_{\pm,A,g_\varepsilon,\ell} = \frac{1}{h}U_{g_\varepsilon,\ell}\left[\theta(4R_\theta^2|p|_{g_0(q)}) + K^h_{\pm,A,g_0}(\mathcal{T}_1)\right]U^*_{g,\ell} \quad , \quad h = 2^{-2\ell},$$

is maximal accretive while the condition (73) ensures

$$\begin{aligned}\forall\ell \leq \ell_{\chi,g_\varepsilon}, \qquad & (u \in D(K_{\pm,A,g_\varepsilon})) \Rightarrow (\chi_\ell u \in D(K_{\pm,A,g_\varepsilon}) \subset D(K_{\pm,A,g_\varepsilon,\ell}))\,,\\ & \forall u \in D(K_{\pm,A,g_\varepsilon})\,, \quad K_{\pm,A,g_\varepsilon}\chi_\ell u = K_{\pm,A,g_\varepsilon,\ell}\chi_\ell u\,.\end{aligned}$$

Moreover, there exists a constant C^1_{χ,g_ε} independent of $\ell \geq \ell_{\chi,g_\varepsilon}$ such that

$$N_{t,g_\varepsilon}(\chi_\ell u\,, \lambda) \leq C^1_{\chi,g_\varepsilon}\|(K_{\pm,A,g_\varepsilon,\ell} - i\lambda)\chi_\ell u\|$$

holds for all $(u, \lambda) \in D(K_{\pm,A,g_\varepsilon}) \times \mathbb{R}$ and all $\ell \geq \ell_{\chi,g_\varepsilon}$.
We now replace $\mathbb{N}$ by $\mathcal{L} = \{-1\}\cup\{\ell \geq \ell_{\chi,g_\varepsilon}\}$ and set $\chi_{-1}^2 = \sum_{\ell=0}^{\ell_{2R_1,g_\varepsilon}-1}\chi_\ell^2$. All the results of Paragraph 6.3.2, namely Lemma 6.7, Lemma 6.8 and Proposition 6.9, still hold with $\mathcal{L} = \mathbb{N}$ replaced by $\mathcal{L} = \{-1\} \cup \{\ell \geq \ell_{\chi,g_\varepsilon}\}$.
Again when $u \in D(K_{\pm,A,g_\varepsilon})$ the condition (73) ensures $\chi_{-1}u \in D(K_{\pm,A,g_\varepsilon})$ while $\mathrm{supp}\,\chi_{-1}u \subset \left\{|p|_{g_\varepsilon(q)} \leq R_1 2^{\ell_{\chi,g_\varepsilon}-1}\right\}$. By Lemma 7.6 applied with $R = R_1 2^{\ell_{\chi,g_\varepsilon}}$, there exists $\mathcal{T}_{-1} = (b, S, T_1, T_2, \theta_{-1})$ and a constant $C^2_{\chi,g_\varepsilon} > 0$ such that

$$C^2_{\chi,g_\varepsilon} + K_{\pm,A,g_\varepsilon,-1} = U_{g_\varepsilon}(C^2_{\chi,g_\varepsilon} + K_{\pm,A,g_0}(\mathcal{T}_{-1}))U^*_{g_\varepsilon}$$

is maximal accretive, and the relations

$$\begin{aligned}&K_{\pm,A,g_\varepsilon}\chi_{-1}u = K_{\pm,A,g_\varepsilon,-1}\chi_{-1}u\,,\\ &N_{t,g_\varepsilon}(\chi_{-1}u, \lambda) \leq C^2_{\chi,g_\varepsilon}\|(K_{\pm,A,g_\varepsilon,-1} + C^2_{\chi,g_\varepsilon} - i\lambda)\chi_{-1}u\|\,,\end{aligned}$$

hold for all $(u, \lambda, t) \in D(K_{\pm,A,g_\varepsilon}) \times \mathbb{R} \times [0, \frac{1}{18})$ when $A \neq 0$ (take $t = \frac{1}{3}$ for the case $A = 0$).
Domain and subelliptic estimate: Take $C \geq C^2_{\chi,g_\varepsilon}$ and $\lambda \in \mathbb{R}$. By definition $u \in D(K_{\pm,A,g_\varepsilon})$ means

$$\begin{aligned}&\|u\|^2_{L^2(Q;\mathcal{H}^1)} + \|(P_{\pm,Q,g_\varepsilon} + C - i\lambda)u\|^2 + \|\gamma_{odd}u\|^2_{L^2(\partial X;|p_1|dq'dp;\mathfrak{f})} < +\infty\,,\\ &\gamma_{ev}u \in L^2_{loc}(\partial X, |p_1|dq'dp; \mathfrak{f}) \quad \gamma_{odd}u = \mathrm{sign}\,(p_1)A(|p|_q)\gamma_{odd}u \text{ for a.e. } |p|_q\,.\end{aligned}$$

With Proposition 6.9 applied with $\sum_{\ell\in\mathcal{L}}\chi_\ell^2\equiv 1$, $\mathcal{L}=\{-1\}\cup\{\ell\geq \ell_{\chi,g_\varepsilon}\}$, and $P_{\pm,Q,g}\chi_\ell u=K_{\pm,Q,g,\ell}u$, it means simply

$$\begin{aligned}
&\forall \ell\in\mathcal{L}\,,\quad \chi_\ell u\in D(K_{\pm,A,g_\varepsilon,\ell})\,,\\
&\sum_{\ell\in\mathcal{L}}\|\chi_\ell u\|^2_{L^2(Q;\mathcal{H}^1)}+\|(K_{\pm,Q,g_\varepsilon}+C-i\lambda)\chi_\ell u\|^2\\
&\qquad\qquad\qquad\qquad\qquad+\|\gamma_{odd}\chi_{\ell u}\|^2_{L^2(\partial X;|p_1|dq'dp;\mathfrak{f})}<+\infty\,.
\end{aligned}$$

Summing the lower bound $\sum_{\ell\in\mathcal{L}}N_{t,g_\varepsilon}(\chi_\ell u,\lambda)^2$ of $\|(K_{\pm,A,g_\varepsilon,\ell}+C-i\lambda)\chi_\ell u\|^2$ after setting, $R_\ell=2R_12^\ell$ for $\ell\geq\ell_{\chi,g_\varepsilon}$ and $R_\ell=R_12^{\ell_{\chi,g_\varepsilon}}$ for $\ell=-1$,

$$\sum_{\ell\in\mathcal{L}}N_{t,g_\varepsilon}(\chi_\ell u,\lambda)^2\leq C^3_{\chi,g_\varepsilon}\left[\|(K_{\pm,Q,g_\varepsilon}+C-i\lambda)u\|^2+\|u\|^2_{L^2(Q;\mathcal{H}^1)}\right]\,.$$

It contains $(1+|p|)^{\tilde{\nu}_3}\gamma u\in L^2(\partial X;|p_1|dq'dp;\mathfrak{f})$ with $\tilde{\nu}_3=-1$ (resp. $\tilde{\nu}_3=0$) when $A=0$ (resp. $A\neq 0$). The summation of $\|\chi_\ell u\|^2_{L^2(Q;\mathcal{H}^1)}$ and $\|\chi_\ell u\|^2_{H^t(\overline{Q};\mathcal{H}^0)}$ are estimated from below by Lemma 6.7 and Lemma 6.8 so that

$$\langle\lambda\rangle^{\frac{1}{4}}\|u\|^2_{L^2(Q;\mathcal{H}^1)}\leq N_{t,g_\varepsilon}(u,\lambda)^2\leq C^3_{\chi,g_\varepsilon}\left[\|(K_{\pm,Q,g_\varepsilon}+C-i\lambda)u\|^2+\|u\|^2_{L^2(Q;\mathcal{H}^1)}\right]\,.$$

Taking λ_0 large enough implies

$$\forall u\in D(K_{\pm,A,g_\varepsilon})\,,\quad N_{t,g_\varepsilon}(u,\lambda)\leq C^4_{\chi,g_\varepsilon}\|(K_{\pm,A,g_\varepsilon}+C-i\lambda)u\|\,,$$

for all $u\in D(K_{\pm,A,g_\varepsilon})$ and all $\lambda\in\mathbb{R}$ such that $|\lambda|\geq\lambda_0$.
Approximation in $D(K_{\pm,A,g_\varepsilon})$: The squared graph norm $\|u\|^2+\|K_{\pm,A,g_\varepsilon}u\|^2$ is equivalent to the series

$$\sum_{\ell\in\mathcal{L}}\|(K_{\pm,Q,g_\varepsilon,\ell}+C-i\lambda)\chi_\ell u\|^2$$

which is the limit of finite sums. Thus the set of $u\in D(K_{\pm,A,g_\varepsilon})$ such that

$$\operatorname{supp}u\subset\left\{g^{ij}_\varepsilon(q)p_ip_j\leq R_u^2\right\}$$

for some $R_u\in(0,+\infty)$, is dense in $D(K_{\pm,A,g_\varepsilon})$ endowed with the graph norm.
Accretivity of $K_{\pm,A,g_\varepsilon}-\frac{d}{2}$: When $u\in D(K_{\pm,A,g_\varepsilon})$ satisfies $\operatorname{supp}u\subset$

$\{g_{\varepsilon}^{ij}(q)p_ip_j \leq R_u^2\}$, Lemma 7.6 with $R = R_u$ implies that u fulfills the inequality (89). Since all the terms of (89) are continuous on $D(K_{\pm,A,g_\varepsilon})$ endowed with the graph norm, it is extended to all $u \in D(K_{\pm,A,g_\varepsilon})$. In particular the subelliptic estimate can be written now with $C = 0$ and any $\lambda \in \mathbb{R}$.

Maximal accretivity: For $C \geq C_\chi^2$ and $\lambda \in \mathbb{R}$, all the operators $K_{\pm,A,g_\varepsilon,\ell} + C - i\lambda$ are invertible. For $f \in L^2(X;\mathfrak{f})$ set $v_\ell = \chi_\ell(K_{\pm,A,g_\varepsilon,\ell} + C - i\lambda)^{-1}\chi_\ell f$. The subelliptic estimates of Lemma 7.6 and Lemma 7.7 imply

$$\|(K_{\pm,A,g_\varepsilon,\ell} + C - i\lambda)^{-1}\chi_\ell f\|_{L^2(Q;\mathcal{H}^1)} \leq C_{\chi,g_\varepsilon}\langle\lambda\rangle^{-\frac{1}{8}}\|\chi_\ell f\|.$$

With $\sup_{\ell'\in\mathcal{L}} \sharp\{\ell \in \mathcal{L}\,, \chi_\ell\chi_{\ell'} \neq 0\} \leq N_\chi$, we deduce that $v = \sum_{\ell\in\mathcal{L}} v_\ell$ satisfies

$$\begin{aligned}
\|v\|_{L^2(Q;\mathcal{H}^1)}^2 &\leq C_\chi \sum_{\ell\in\mathcal{L}} \|\chi_\ell(K_{\pm,A,g_\varepsilon,\ell} + C - i\lambda)^{-1}\chi_\ell f\|_{L^2(Q;\mathcal{H}^1)}^2 \\
&\leq C'_{\chi,g_\varepsilon} \sum_{\ell\in\mathcal{L}} \|\chi_\ell f\|^2 = C_{\chi,g_\varepsilon}\|f\|^2 . \\
\gamma v = \sum_{\ell\in\mathcal{L}} \gamma v_\ell &\in L^2_{loc}(\partial X, |p_1|dq'dp;\mathfrak{f}) , \\
\gamma_{odd} v &= \sum_{\ell} \chi_\ell \gamma_{odd}(K_{\pm,A,g_\varepsilon,\ell} + C - i\lambda)^{-1}\chi_\ell f \\
&= \pm \sum_{\ell\in\mathcal{L}} \chi_\ell\,\mathrm{sign}\,(p_1)A(|p|_q)\gamma_{odd}(K_{\pm,A,g_\varepsilon,\ell} + C - i\lambda)^{-1}\chi_\ell f \\
&= \pm\,\mathrm{sign}\,(p_1)A(|p|_q)\gamma_{ev} v ,
\end{aligned}$$

by using the commutation $[\chi_\ell, A] = 0$ for the last line.
For any $\ell \in \mathcal{L}$, the function $v_\ell \in L^2(Q;\mathcal{H}^1)$ satisfies

$$\begin{aligned}
(P_{\pm,A,g_\varepsilon} + C - i\lambda)v_\ell &= (K_{\pm,A,g_\varepsilon,\ell} + C - i\lambda)v_\ell \\
&= \chi_\ell^2 f - \frac{1}{2}\left[\Delta_p, \chi_\ell\right](K_{\pm,A,g_\varepsilon,\ell} + C - i\lambda)^{-1}\chi_\ell f \\
&= \chi_\ell^2 f + B_\ell(K_{\pm,A,g_\varepsilon,\ell} + C - i\lambda)^{-1}\chi_\ell f ,
\end{aligned}$$

after setting $B_\ell = -\frac{1}{2}\left[\Delta_p, \chi_\ell\right] = -\left[(\nabla_p\chi_\ell).\nabla_p + \frac{(\Delta_p\chi_\ell)}{2}\right]$. With the notations of Paragraph 6.3.2, the operators B_ℓ are bounded in $\mathrm{DiffP}^{1,-1}$, uniformly w.r.t $\ell \in \mathcal{L}$, with $p - \mathrm{supp}\, B_\ell \subset \left\{R_{\chi,g_\varepsilon}^{-1}2^\ell \leq |p|_{g_\varepsilon(q)} \leq R_{\chi,g_\varepsilon}2^\ell\right\}$.
We get like in Proposition 6.9

$$\|\sum_{\ell\in\mathcal{L}} B_\ell(K_{\pm,A,g_\varepsilon,\ell} + C - i\lambda)\chi_\ell f\|^2 \leq C_{\chi,g_\varepsilon}^7\langle\lambda\rangle^{-\frac{1}{4}}\|f\|^2 .$$

For any $f \in L^2(X;\mathfrak{f})$ we have found $v = \sum_{\ell\in\mathcal{L}} v_\ell \in D(K_{\pm,A,g_\varepsilon})$ such that

$$(K_{\pm,A,g_\varepsilon} + C - i\lambda)v = (\mathrm{Id} + B)f$$

with $\|B\|_{\mathcal{L}(L^2(X;\mathfrak{f}))} \leq \sqrt{C^7_{\chi,g_\varepsilon}} \langle\lambda\rangle^{-\frac{1}{8}}$. For λ large enough $(\mathrm{Id} + B)$ is invertible and $(K_{\pm,A,g_\varepsilon} + C - i\lambda)$ is invertible. □

7.4.2 Estimate of $\|\Phi(q^1)\mathcal{O}_{Q,g_\varepsilon}u\|$

In the definition (90) estimated in Proposition 7.5, the quantity $\|\Phi(q^1)\mathcal{O}_{Q,g_\varepsilon}u\|$ is missing. Actually our approach which handles non maximal subelliptic estimates with exponent divided by 2 in the end does not provide directly an upper bound for it. It can be obtained in a second step by adapting the prof of Proposition 5.10.

Proposition 7.8. *Let $K_{\pm,A,g_\varepsilon}$ be the maximal accretive realization of $P_{\pm,Q,g_\varepsilon}$ defined in Proposition 7.5 with $\overline{Q} = (-\infty,0] \times \mathbb{T}^{d-1}$, $\varepsilon \leq \varepsilon_g$ small enough and A satisfying Hypothesis 4 and (73). For any $\Phi \in \mathcal{C}^\infty_b((-\infty,0])$ such that $\Phi(0) = 0$. There exists a constant C_{g_ε} independent of Φ and a constant $C_{g_\varepsilon,\Phi}$ such that*

$$\|\Phi(q^1)\mathcal{O}_{Q,g_\varepsilon}u\| \leq C_{g_\varepsilon}\|\Phi\|_{L^\infty}\|(K_{\pm,A,g_\varepsilon} - i\lambda)u\| + C_{g_\varepsilon,\Phi}\|u\|\,,$$

holds for all $u \in D(K_{\pm,A,g_\varepsilon})$ and all $\lambda \in \mathbb{R}$.

Proof. We embed the manifold $\overline{Q} = (-\infty,0] \times Q'$, $Q' = \mathbb{T}^{d-1}$, into $\tilde{Q} = \mathbb{R} \times Q'$. The metric $\tilde{g}_\varepsilon$ on $\tilde{Q}$ is assumed to be $\mathcal{C}^\infty$ with $\tilde{g}_\varepsilon\big|_{\overline{Q}} = g_\varepsilon$ and $\tilde{g}_\varepsilon - g_0 \in \mathcal{C}^\infty_0(\tilde{Q})$. Due to the curvature of ∂Q the metric $\tilde{g}_\varepsilon$ is not given by a simple symmetry argument but a $\mathcal{C}^\infty$ extension is always possible, first locally (see [ChPi]) and then globally with a partition of unity.
When $u \in D(K_{\pm,A,g_\varepsilon})$ and $\Phi(0) = 0$, the function $\Phi(q^1)u$ belongs to $D(K_{\pm,\tilde{Q},\tilde{g}_\varepsilon})$ and Proposition 6.10 about whole cylinders provides

$$\|\mathcal{O}_{\tilde{Q},\tilde{g}_\varepsilon}\Phi(q^1)u\| \leq C_{g_\varepsilon}\|(K_{\pm,\tilde{Q},\tilde{g}_\varepsilon} - i\lambda)\Phi(q^1)u\|\,,$$

equivalently written

$$\|\Phi(q^1)\mathcal{O}_{Q,g_\varepsilon}u\| \leq C_{g_\varepsilon}\|(K_{\pm,Q,g_\varepsilon} - i\lambda)\Phi(q^1)u\|\,.$$

We compute

$$(K_{\pm,Q,g_\varepsilon} - i\lambda)(\Phi(q^1))u) = \Phi(q^1)(K_{Q,A,g_\varepsilon} - i\lambda)u + p_1\Phi'(q^1)u\,,$$

in order to get

$$\|\Phi(q^1)\mathcal{O}_{Q,g_\varepsilon}u\| \le C^1_{g_\varepsilon}\|\Phi(q^1)(K_{\pm,Q,g_\varepsilon}-i\lambda)u\| + C^1_{\Phi,g_\varepsilon}\|u\|_{L^2(Q;\mathcal{H}^1)}\,.$$

But the integration by part identity (89) and the accretivity of $K_{\pm,A,g_\varepsilon}-\frac{d}{2}$

$$\begin{aligned}\|u\|^2_{L^2(Q;\mathcal{H}^1)} &\le \|u\|\|(K_{\pm,A,g_\varepsilon}+\frac{d}{2}-i\lambda)u\|\\ &\qquad 2\|u\|\|(K_{\pm,A,g_\varepsilon}-i\lambda)u\| \le \left[\delta\|(K_{\pm,A,g_\varepsilon}-i\lambda)u\|+\delta^{-1}\|u\|\right]^2\,,\end{aligned}$$

for all $\delta>0$. Choosing $\delta=\frac{C^1_{g_\varepsilon}\|\Phi\|_{L^\infty}}{C^1_{\Phi,g_\varepsilon}}$ leads to

$$\|\Phi(q^1)\mathcal{O}_{Q,g_\varepsilon}u\| \le 2C^1_{g_\varepsilon}\|\Phi\|_{L^\infty}\|(K_{\pm,A,g_\varepsilon}-i\lambda)u\| + C^2_{\Phi,g_\varepsilon}\|u\|\,,$$

with $C^2_{\Phi,g_\varepsilon}=\frac{(C^1_{\Phi,g_\varepsilon})^2}{C^1_{g_\varepsilon}\|\Phi\|_{L^\infty}}$. □

7.4.3 Adjoint

Proposition 7.9. *Let $K_{\pm,A,g_\varepsilon}$ be the maximal accretive realization of $P_{\pm,Q,g_\varepsilon}$ defined in Proposition 7.5 with $\overline{Q}=(-\infty,0]\times\mathbb{T}^{d-1}$, $\varepsilon\le\varepsilon_g$ small enough and A satisfying Hypothesis 4 and* (73). *The adjoint of $K_{\pm,A,g_\varepsilon}$ is $K_{\mp,A^*,g_\varepsilon}$.*

Proof. The adjoint A^* of A fulfills Hypothesis 4 and (73). Thus $K_{\mp,A^*,g_\varepsilon}$ is maximal accretive and shares the same properties as $K_{\pm,A,g_\varepsilon}$ while replacing A with A^*, $\pm$ with $\mp$. Take first $u\in D(K_{\mp,A^*,g_\varepsilon})$ and $v\in D(K_{\pm,A,g_\varepsilon})$ such that $\operatorname{supp}u\subset B_{R_u}$ and $\operatorname{supp}v\subset B_{R_v}$, where we recall

$$B_R=\left\{g^{ij}_\varepsilon(q)p_ip_j\le R^2\right\}\,.$$

By applying Lemma 7.6 with $R=\max\{R_u,R_v\}$ there exists $\mathcal{T}=(b,S,T_1,T_2,\theta)$ such that

$$\begin{aligned}
K_{\mp,A^*,g_\varepsilon} &= U_{g_\varepsilon}K^1_{\mp,A^*,g_0}(\mathcal{T})U^*_{g_\varepsilon}\\
K^1_{\mp,A^*,g_0}(\mathcal{T}) &= K_{\mp,A^*,g_0}\mp p_ib^i(q)\theta(|p|_{g_0(q)})\mp S^{ij}_k(q)\theta(|p|_{g_0(q)})p_ip_j\partial_{p_k}\\
&\qquad +\frac{-\partial_p^TT_1(q)\partial_p+p^TT_2(q)p}{2}\,,\\
K_{\pm,A,g_\varepsilon} &= U_{g_\varepsilon}K^1_{\pm,A,g_0}(\mathcal{T})U^*_{g_\varepsilon}\\
K^1_{\pm,A,g_0}(\mathcal{T}) &= K_{\pm,A,g_0}\pm p_ib^i(q)\theta(|p|_{g_0(q)})\pm S^{ij}_k(q)\theta(|p|_{g_0(q)})p_ip_j\partial_{p_k}\\
&\qquad +\frac{-\partial_p^TT_1(q)\partial_p+p^TT_2(q)p}{2}\,.
\end{aligned}$$

By setting $\tilde{u} = U_{g_\varepsilon} u \in D(K^1_{\mp,A^*,g_0}(\mathcal{T})) = D(K_{\mp,A^*,g_0})$ and $\tilde{v} = U_{g_\varepsilon} v \in D(K^1_{\pm,A,g_0}(\mathcal{T}) = D(K_{\pm,A,g_0}))$, the equality $K^*_{\pm,A,g_0} = K_{\mp,A^*,g_0}$ given in Proposition 7.1 leads to

$$\begin{aligned}\langle K_{\mp,A^*,g_\varepsilon} u\,, v\rangle - \langle u\,,\, K_{\pm,A,g_\varepsilon} v\rangle &= \langle K^1_{\mp,A^*,g_0}(\mathcal{T})\tilde{u}\,,\, \tilde{v}\rangle - \langle \tilde{u}\,,\, K^1_{\pm,A,g_\varepsilon}(\mathcal{T})\tilde{v}\rangle \\ &= \mp 2\langle \tilde{u}\,,\, p_i b^i(q)\tilde{v}\rangle \pm \langle \tilde{u}\,,\, [p_i S^{ij}_j(q) + p_j S^{ij}_i(q)]\tilde{v}\rangle\,,\end{aligned}$$

where we used $\theta(|p|_{g_0(q)}) \equiv 1$ in a neighborhood of $\operatorname{supp}\tilde{u} \cup \operatorname{supp}\tilde{v}$.
The right-hand side is continuous on $L^2(Q;\mathcal{H}^1)$. It vanishes when $u, v \in \mathcal{C}^\infty_0(X;\mathfrak{f})$ by direct calculations, and therefore when $\tilde{u}\,,\tilde{v} \in \mathcal{C}^\infty_0(X;\mathfrak{f})$ with $\theta \equiv 1$ in a neighborhood of $\operatorname{supp}\tilde{u} \cup \operatorname{supp}\tilde{v}$. By density it always vanishes.
For $u \in D(K_{\mp,A^*,g_\varepsilon})$ such that $\operatorname{supp} u \subset B_{R_u}$, $R_u < +\infty$, we have proved

$$\langle K_{\mp,A^*,g_\varepsilon} u\,,\, v\rangle = \langle u\,,\, K_{\pm,A,g_\varepsilon} v\rangle$$

for all $v \in D(K_{\pm,A,g_\varepsilon})$ which fulfill $\operatorname{supp} v \subset B_{R_v}$ for some $R_v < +\infty$. Those v's are dense in $D(K_{\pm,A,g_\varepsilon})$ endowed with the graph norm. Therefore $u \in D(K^*_{\pm,A,g_\varepsilon})$. All those u's are dense in $D(K_{\mp,A^*,g_\varepsilon})$ and we obtain $K_{\mp,A^*,g_\varepsilon} \subset K^*_{\pm,A,g_\varepsilon}$. Both operators are maximal accretive and this yields the equality. $\square$

7.4.4 Density of $\mathcal{D}(\overline{X}, j)$ in $D(K_{\pm,0,g_\varepsilon})$

When $\overline{Q} = (-\infty, 0] \times \mathbb{T}^{d-1}$ we recall that $u \in \mathcal{D}(\overline{X}, j)$ is characterized by

$$\begin{aligned}&u \in \mathcal{C}^\infty_0(\overline{X};\mathfrak{f}) \quad , \quad \gamma_{odd} u = 0\,, i.e.\ \gamma u(q', -p_1, p') = j\gamma u(q', p_1, p') \\ &\partial_{q^1} u = \mathcal{O}(|q^1|^\infty)\,.\end{aligned}$$

Proposition 7.10. *Within the framework of Proposition 7.5 and in the case $A = 0$, $\mathcal{D}(\overline{X}, j)$ is dense in $D(K_{\pm,0,g_\varepsilon})$ endowed with its graph norm.*

Proof. Remember the notation

$$B_R = \left\{(q,p) \in \overline{X}\,, g^{ij}_\varepsilon(q) p_i p_j \leq R\right\}$$

By Proposition 7.5 the set

$$\{u \in D(K_{\pm,0,g_\varepsilon})\,, \exists R_u > 0\,, \operatorname{supp} u \subset B_{R_u}\}$$

is dense in $D(K_{\pm,0,g_\varepsilon})$.
By Lemma 7.6 the unitary change of variables (91)(92),

$$\begin{aligned}&(U_{g_\varepsilon}v)(q,p) = \det(\Psi(q))^{-1/2}v(q,\Psi(q)^{-1}p)\,,\\ &\Psi(q) = g_\varepsilon(q)g_0(q)^{-1}\,,\end{aligned}$$

allows to write

$$K_{\pm,0,g_\varepsilon}u = U_{g_\varepsilon}K^1_{\pm,0,g_0}(\mathcal{T})U^*_{g\varepsilon}u$$

for some $\mathcal{T} = (b, S, T_1, T_2, \theta)$ fixed by $R > 0$ as soon as $\operatorname{supp} u \subset B_R$. But $\mathcal{D}(\overline{X}, j)$ is dense in $D(K_{\pm,0,g_0}) = D(K_{\pm,0,g_0}(\mathcal{T}))$. Since the unitary transform preserves the set of $u \in \mathcal{C}_0^\infty(\overline{X};\mathfrak{f})$ such that $\gamma_{odd}u = 0$, we deduce that this set is dense in $D(K_{\pm,0,g_\varepsilon})$ endowed with its graph norm.
Let $u \in \mathcal{C}_0^\infty(\overline{X};\mathfrak{f})$ satisfy $\gamma_{odd}u = 0$. We now prove that u can be approximated in $D(K_{\pm,0,g_\varepsilon})$ by elements of $\mathcal{D}(\overline{X}, j)$. By Hadarmard's lemma (or Taylor expansion with integral remainder), there exists a cut-off function $\chi \in \mathcal{C}_0^\infty((-\infty, 0])$, $\chi \equiv 1$ in a neighborhood of 0, and $v \in \mathcal{C}_0^\infty(\overline{X},\mathfrak{f})$ such that

$$u(q,p) = \chi(q^1)u(0,q',p) + q^1v(q,p)\,.$$

We take for $n \in \mathbb{N}^*$,

$$u_n = u + (1-\chi(nq^1))q^1v(q,p) = u - \chi(nq^1)q^1v(q,p)\,.$$

Clearly $u_n \in \mathcal{D}(\overline{X}, j)$ for all $n \in \mathbb{N}^*$ $\lim_{n\to\infty}\|u_n - u\| = 0$ and

$$P_{\pm,g_\varepsilon}u_n = P_{\pm,g_\varepsilon}u - \chi(nq^1)q^1P_{\pm,g_\varepsilon}v - p_1\left[\chi(nq^1) + \chi'(nq^1)nq^1\right]v$$

converges to $P_{\pm,g_\varepsilon}u$ in $L^2(X, dqdp;\mathfrak{f})$. $\square$

7.5 Global result

We now end the proof of Theorem 1.1 and Theorem 1.2 by using a partition of unity in q. For this we assume $A = A(q, |p|_q)$ like in (5)(6)(7)(8) or (77)(78)(79). When $\overline{Q} = Q \sqcup \partial Q$ is a compact manifold or a compact perturbation of the euclidean half-space $\overline{\mathbb{R}^d_-}$. Every interior chart domain diffeomorphic to a bounded domain of $(-1,1)^d$ can be embedded in $\mathbb{T}^d$, while every boundary chart domain, diffeomorphic to a bounded domain of $(-\infty, 0] \times (-1,1)^{d-1}$ can be embedded in the half-cylinder $(-\infty, 0] \times \mathbb{T}^{d-1}$.

For every $q \in \partial Q$ one can find a neighborhood $\mathcal{U}$ of q, embedded in $(-\infty, 0] \times \mathbb{T}^{d-1}$, a metric $g_{\mathcal{U},\varepsilon}$ of the form

$$g_{\mathcal{U},\varepsilon} = \begin{pmatrix} 1 & 0 \\ 0 & m_{\mathcal{U},\varepsilon}(q^1, q') \end{pmatrix}$$
$$m_{,\mathcal{U},\varepsilon}(q^1, q') = m_{\mathcal{U},0}(q') + \chi_m(\frac{q^1}{\varepsilon})q^1 \tilde{m}_{\mathcal{U}}(q^1, q'),$$

coincide with g on $\mathcal{U}$. Furthermore $\mathcal{U}$ can be chosen small enough so that the condition $\varepsilon \leq \varepsilon_{g_{\mathcal{U}}}$ of Proposition 7.5 is satisfied. By compactness, there exists an (at most countable) locally finite covering $\overline{Q} = \cup_{\ell \in \mathcal{L}} \mathcal{U}_\ell$ so that the condition $\varepsilon \leq \varepsilon_{g_{\mathcal{U}}}$ is satisfies for all $\mathcal{U}$ such that $\mathcal{U} \cap \partial Q \neq \emptyset$. The partition of unity $(\chi_\ell(q))_{\ell \in \mathcal{L}}$ such that $\sum_{\ell \in \mathcal{L}} \chi_\ell^2(q) \equiv 1$ is subordinate to this covering. This partition of unity can be chosen so that

$$\mathrm{supp}_{\ell' \in \mathcal{L}} \sharp \{\ell \in \mathcal{L}, \chi_\ell \chi_{\ell'} \neq 0\} \leq N_\chi < +\infty .$$

Proof of Theorem 1.1 and Theorem 1.2: Let $(\chi_\ell(q))_{\ell \in \mathcal{L}}$ be the above partition of unity.
We recall that $u \in D(K_{\pm,A,g})$ is characterized by

$$\begin{aligned}
& u \in L^2(Q; \mathcal{H}^1) \quad , \quad P_{\pm,Q,g} u \in L^2(X, dqdp; \mathfrak{f}), \\
& \forall R > 0 , 1_{[0,R]}(|p|_q) \gamma u \in L^2(\partial X, |p_1| dq' dp; \mathfrak{f}), \\
& \gamma_{odd} u = \pm \,\mathrm{sign}\,(p_1) A \gamma_{ev} u .
\end{aligned}$$

The additional condition $A = A(q, |p|_q)$ and this definition of $D(K_{\pm,A,g})$ implies that for any $\ell \in \mathcal{L}$,

$$(u \in D(K_{A,g})) \to (\chi_\ell u \in D(K_{A,g})) .$$

Moreover with $\chi_\ell = \chi_\ell(q)$, one gets

$$\mathrm{ad}_{\chi_\ell} K_{\pm,A,g} = g^{ij}(q) p_i \partial_{q^j} \chi_\ell \quad , \quad \mathrm{ad}^2_{\chi_\ell} K_{\pm,A,g} = 0 .$$

Like in Subsection 6.3.1, we infer the equivalences

$$\left(\frac{\|u\|_{H^t(\overline{Q};\mathcal{H}^0)}}{\sum_{\ell \in \mathcal{L}} \|\chi_\ell u\|_{H^t(\overline{Q};\mathcal{H}^0)}} \right) \leq C_t$$
$$\text{and} \qquad \left(\frac{\|(K_{\pm,A,g} - i\lambda)u\|^2 + \|u\|^2_{L^2(Q;\mathcal{H}^1)}}{\sum_{\ell=1}^{L_\varepsilon} \|(K_{\pm,A,g} - i\lambda)\chi_\ell u\|^2 + \|\chi_\ell u\|^2_{L^2(Q;\mathcal{H}^1)}} \right)^{\pm 1} \leq C_\chi ,$$

while the squared norms $\|u\|^2_{L^2(Q;\mathcal{H}^s)}$ and $\|\gamma u\|^2_{L^2(\partial X,|p_1|dq'dp;\mathfrak{f})}$ are equal to

$$\begin{aligned}\|u\|^2_{L^2(Q;\mathcal{H}^s)} &= \sum_{\ell\in\mathcal{L}}\|\chi_\ell u\|^2_{L^2(Q;\mathcal{H}^s)}\\ \|\gamma u\|^2_{L^2(\partial X,|p_1|dq'dp;\mathfrak{f})} &= \sum_{\ell\in\mathcal{L}}\|\gamma\chi_\ell u\|^2_{L^2(\partial X,|p_1|dq'dp;\mathfrak{f})}\,.\end{aligned}$$

Hence u belongs to $D(K_{\pm,A,g})$ if and only if all the $\chi_\ell u$'s belong to $D(K_{\pm,A,g})$ and

$$\sum_{\ell\in\mathcal{L}}\|K_{\pm,A,g}\chi_\ell u\|^2+\|\chi_\ell u\|^2_{L^2(Q,\mathcal{H}^1)}<+\infty\,.$$

This proves that the set $u\in D(K_{\pm,A,g})$ with a compact q-support are dense in $D(K_{\pm,A,g})$ endowed with its graph norm. In particular when $A=0$, the results of Proposition 7.10 and Corollary 6.5 imply that $\mathcal{D}(\overline{X},j)$ is dense in $D(K_{\pm,0,g})$ when $A=0$.
According to (131), the identity $\mathrm{ad}^{\,2}_{\chi_\ell}K_{\pm,A,g}=0$ implies

$$\mathbb{R}\mathrm{e}\,\langle u\,,\,K_{\pm,A,g}u\rangle=\sum_{\ell\in\mathcal{L}}\mathbb{R}\mathrm{e}\,\langle\chi_\ell u\,,\,K_{\pm,A,g}\chi_\ell u\rangle\,,$$

for all $u\in D(K_{\pm,A,g})$. This proves the accretivity of $K_{\pm,A,g}-\frac{d}{2}$ and the integration by part identity

$$\mathbb{R}\mathrm{e}\,\langle u\,K_{\pm,A,g}u\rangle=\|u\|^2_{L^2(Q;\mathcal{H}^1)}+\mathbb{R}\mathrm{e}\,\langle\gamma_{ev}u\,,\,A\gamma_{ev}u\rangle_{L^2(\partial X,|p_1|dq'dp;\mathfrak{f})}\,,$$

for all $u\in D(K_{\pm,A,g})$.
Since the subelliptic estimates of Proposition 7.5 when $\mathrm{supp}\,\chi_\ell\cap\partial Q\neq\emptyset$ and Corollary 6.5 are uniformly satisfied for the local models of $K_{\pm,A,g}\chi$, the above equivalence of norms imply the subelliptic estimates of Theorem 1.1 and Theorem 1.2 by simple summation over $\mathcal{L}$.
The maximal accretivity is checked like in the final proof of Proposition 7.5. For $f\in L^2(X,dqdp;\mathfrak{f})$ take $u=\sum_{\ell\in\mathcal{L}}v_\ell$ with

$$v_\ell=\chi_\ell(K_{\pm,A,g}-i\lambda)^{-1}\chi_\ell f\,.$$

The function u belongs to $D(K_{\pm,A,g})$ and satisfies

$$(K_{\pm,A,g}-i\lambda)u=f+\sum_{\ell\in\mathcal{L}}[g^{ij}(q)p_i\partial_{q^j}\chi_\ell(q)](K_{\pm,A,g}-i\lambda)^{-1}\chi_\ell f=(\mathrm{Id}+B)f\,.$$

Owing to $\mathrm{supp}_{\ell'\in\mathcal{L}}\sharp\{\ell\in\mathcal{L},\chi_\ell\chi_{\ell'}\neq 0\}\leq N_\chi$, we infer from the subelliptic estimates for $(K_{\pm,A,g}-i\lambda)$

$$\|B\|_{\mathcal{L}(L^2(X,dqdp;\mathfrak{f}))}\leq C\langle\lambda\rangle^{-\frac{1}{8}}\,.$$

Hence $\mathrm{Id}+B$ is invertible when λ is chosen large enough and $(K_{\pm,A,g}-i\lambda)$ is invertible. This proves the maximal accretivity.
For $u\in D(K_{\pm,A,g})$ and $v\in D(K_{\mp,A^*,g})$ we compute

$$\begin{aligned}\langle v\,,K_{\pm,A,g}u\rangle-\langle K_{\mp,A^*,g}v\,,u\rangle-\sum_{\ell\in\mathcal{L}}\langle\chi_\ell v\,,K_{\pm,A,g}\chi_\ell u\rangle-\langle\chi_\ell K_{\mp,A^*,g}v\,,\chi_\ell u\rangle\\ =\langle v\,,\,(P_{\pm,Q,g}-\sum_{\ell\in\mathcal{L}}\chi_\ell P_{\pm,Q,g}\chi_\ell)u\rangle-\langle(P_{\mp,Q,g}-\sum_{\ell\mathcal{L}}\chi_\ell P_{\mp,Q,g}\chi_\ell)v\,,\,u\rangle\\ =\sum_{\ell\in\mathcal{L}}\langle v\,,\,\chi_\ell g^{ij}(q)p_i(\partial_{q^j}\chi_\ell)u\rangle+\langle\chi_\ell g^{ij}(q)p_i(\partial_{q^j}\chi_\ell)v\,,\,u\rangle=0\,.\end{aligned}$$

But every term $\langle\chi_\ell v\,,\,K_{\pm,A,g}\chi_\ell u\rangle-\langle K_{\mp,A^*,g}\chi_\ell v\,,\,\chi_\ell u\rangle$ vanishes. We have proved

$$\forall u\in D(K_{\pm,A,g})\,,\,\forall v\in D(K_{\mp,A^*,g})\,,\langle v\,,\,K_{\pm,A,g}u\rangle=\langle K_{\mp,A^*,g}v\,,\,u\rangle$$

and the adjoint of $K_{\pm,A,g}$ equals $K_{\mp,A^*,g}$. □

8 Variations on a Theorem

In this section, some straightforward consequences and variants of Theorem 1.1 and Theorem 1.2 are listed.

8.1 Corollaries

We refer to the definitions and results of Section 3.

Corollary 8.1. *Within the framework of Theorem 1.1 and Theorem 1.2 the operator $K_{\pm,A,g}$ is $\frac{1}{4}$-pseudospectral. Its spectrum is contained in $S\cap\left\{\mathrm{Re}\;z\geq\frac{d}{2}\right\}$ with*

$$S=\left\{z\in\mathbb{C}\,,|z+1|\leq C(\mathrm{Re}\;z+1)^4\,,\mathrm{Re}\;z\geq -1\right\}\,,$$

and the resolvent estimate

$$\forall z\notin S\,,\quad\|(z-K_{\pm,A,g})^{-1}\|\leq C\langle z\rangle^{-\frac{1}{4}}\,.$$

The constant $C > 0$ can be chosen so that the semigroup $(e^{-tK_{\pm,A,g}})_{t\geq 0}$ is given by the convergent contour integral

$$e^{-tK_{\pm,A,g}} = \frac{1}{2\pi i}\int_{\partial S} e^{-tz}(z - K_{\pm,A,g})\, dz \quad , \quad t > 0\,,$$

with ∂S oriented from $+i\infty$ to $-i\infty$.
It satisfies the estimate

$$\sup_{t>0} \|t^7 K_{\pm,A,g} e^{-tK_{\pm,A,g}}\| < +\infty\,.$$

The above contour integral easily implies exponential decay estimate under a spectral gap condition.

Corollary 8.2. *Within the framework of Theorem 1.1 and Theorem 1.2, assume that the spectrum $\sigma(K_{\pm,A,g})$ is partitioned into to parts*

$$\begin{array}{ll} & \sigma(K_{\pm,A,g}) \subset \sigma_0 \cup \sigma_\infty \\ \text{with} & \sigma_0 \subset \{z \in \mathbb{C}\,,\ \mathrm{Re}\ z \leq \mu_0\} \quad , \quad \sigma_\infty \subset \{z \in \mathbb{C}\,, \mathrm{Re}\ z \geq \mu\} \quad , \quad \mu > \mu_0\,. \end{array}$$

Let Π_0 be the spectral projection associated with σ_0. For any $\tau < \mu$ there exists a constant $C_\tau > 0$ such that

$$\forall t \geq 1\,, \quad \|e^{-tK_{\pm,A,g}} - e^{-tK_{\pm,A,g}}\Pi_0\| \leq C_\tau e^{-\tau t}\,.$$

Corollary 8.3. *Within the framework of Theorem 1.1 and Theorem 1.2 and when $\overline{Q}$ is compact, the resolvent of $K_{\pm,A,g}$ is compact and its spectrum $\sigma(K_{\pm,A,g})$ is discrete.*

8.2 PT-symmetry

While studying accurately the spectrum of scalar Kramers-Fokker-Planck operators on $Q = \mathbb{R}^d$, Hérau-Hitrik-Sjöstrand in [HHS2] used the following version of PT-symmetry: When $Q = \mathbb{R}^d$, the operator K satisfies

$$UKU^* = K^* \quad \text{with} \quad Uu(q,p) = u(q,-p) \quad (U* = U)\,. \tag{93}$$

We keep the same notation U for the unitary action on $L^2(\partial X, |p_1|dq'dp; \mathfrak{f})$.

Proposition 8.4. *Within the framework of Theorem 1.1 and Theorem 1.2 assume additionally that A satisfies $UAU^* = A^*$. Then $K_{\pm,A,g}$ is PT-symmetric according to* (93) *and its spectrum is symmetric with respect to the real axis,*

$$\overline{\sigma(K_{\pm,A,g})} = \sigma(K_{\mp,A^*,g}) = \sigma(K_{\pm,A,g})\,.$$

Proof. The relation $UP_{\pm,Q,g}U^* = P_{\mp,Q,g}$ is straightforward and U preserves the conditions

$$u \in L^2(Q;\mathcal{H}^1) \quad , \quad P_{\pm,Q,g}u \in L^2(X, dqdp; \mathfrak{f})\,,$$
$$\forall R > 0\,, 1_{[0,R]}(|p|_q)\gamma u \in L^2(\partial X, |p_1|dq'dp; \mathfrak{f})\,,$$

which occurs the definition of $D(K_{\pm,A,g})$ and of $D(K_{\mp,A^*,g})$. With

$$\begin{aligned}
\gamma_{odd}(U^*u)(q',p) &= \frac{\gamma(U^*u)(q',p_1,p') - j\gamma(U^*u)(q',-p_1,p')}{2} \\
&= \frac{\gamma u(q',-p_1,-p') - j\gamma u(q',p_1,-p')}{2} = (U^*\gamma_{odd}u)(q',p) \\
\gamma_{ev}(U^*u)(q',p) &= \frac{\gamma(U^*u)(q',p_1,p') + j\gamma(U^*u)(q',-p_1,p')}{2} \\
&= \frac{\gamma u(q',-p_1,-p') + j\gamma u(q',p_1,-p')}{2} = (U^*\gamma_{ev}u)(q',p)
\end{aligned}$$

and $$U\mathrm{sign}(p_1)U^* = -\mathrm{sign}(p_1)$$

the condition $UAU^* = A^*$ leads to

$$[\gamma_{odd}(U^*u)] = \mp\,\mathrm{sign}\,(p_1)A^*\gamma_{ev}(U^*u)\,,$$

when $u \in D(K_{\pm,A,g})$.
Obviously, $u \in D(K_{\pm,A,g})$ is equivalent to $U^*u \in D(K_{\mp,A^*,g}) = D(K^*_{\pm,A,g})$ and this ends the proof. ☐

8.3 Adding a potential

When we add a potential the energy is $\mathcal{E}_V(q,p) = \frac{|p|_q^2}{2} + V(q)$ and the Hamiltonian vector field is

$$\mathcal{Y}_{\mathcal{E}_V} = \mathcal{Y}_{\mathcal{E}} - \partial_{q^i}V(q)\partial_{p_i}\,.$$

The corresponding Kramers-Fokker-Planck operator is

$$P_{\pm,Q,g}(V) = P_{\pm,Q,g} \mp \partial_{q^i}V(q)\partial_{p_i}\,.$$

Proposition 8.5. *When V is a globally Lipschitz function on $\overline{Q}$, the operator $K_{\pm,A,g}(V)-\frac{d}{2}=K_{\pm,A,g}-\frac{d}{2}\mp\partial_{q^i}V(q)\partial_{p_i}$ with the domain $D(K_{\pm,A,g}(V))=D(K_{\pm,A,g})$ is maximal accretive and shares the same properties as $K_{\pm,A,g}$ by simply changing the constants in the subelliptic estimates.*
Its adjoint is $K_{\mp,A^*,g}(V)$.
When $\overline{Q}$ is compact, the resolvent of $K_{\pm,A,g}$ is compact and its spectrum is discrete.
If $UAU^*=A^*$ with $Uu(q,p)=u(q,-p)$, then $K_{\pm,A,g}(V)$ satisfies the PT-symmetry property (93) *and*

$$\sigma(K_{\pm,A,g}(V))=\overline{\sigma(K_{\mp,A^*,g}(V))}=\overline{\sigma(K_{\pm,A,g}(V))}\,.$$

Proof. The subelliptic estimates of Theorem 1.1 and Theorem 1.2 include

$$\forall u\in D(K_{\pm,A,g})\,,\langle\lambda\rangle^{\frac{1}{8}}\|u\|_{L^2(Q;\mathcal{H}^1)}\leq C\|(K_{\pm,A,g}-i\lambda)u\|\,.$$

Combined with Proposition 3.9 applied with $K=C+K_{\pm,A,g}$ and $C>0$ large enough and Corollary 3.10, the inequality

$$\|\partial_{q^i}V(q)\partial_{p_i}u\|\leq\|\partial_qV\|_{L^\infty}\|u\|_{L^2(Q;\mathcal{H}^1)}$$

yields the result. $\square$

Remark 8.6. *Of course all the consequences listed in Subsection 8.1 are valid.*

Remark 8.7. *The hamiltonian flow is not well defined under the sole assumption that V is Lipschitz continuous. It is not a surprise that the dynamics (the semigroup) is well defined as soon as the diffusion term $\mathcal{O}_{Q,g}$ is added. It was already observed in [HelNi] that the Kramers-Fokker-Planck operator on $\mathbb{R}^{2d}$ with any $\mathcal{C}^\infty$ potential $V\in\mathcal{C}^\infty(\mathbb{R}^d)$ is essentially maximal accretive on $\mathcal{C}_0^\infty(\mathbb{R}^{2d})$, although the hamiltonian flow is not always well defined.*

8.4 Fiber bundle version

The final proof of Theorem 1.1 and Theorem 1.2 reduces the problem to local ones via spatial partition of unities. There is therefore no difficulties to replace $\overline{Q}\times\mathfrak{f}$ by some Hermitian bundle. More precisely we assume that $\pi_F:F\to\overline{Q}$ is a smooth finite dimensional Hermitian bundle, with typical fiber $F_q \sim \mathfrak{f}$, endowed with a connection ∇^F which is an End(F)-valued 1-form on TQ. Locally a section of F, $s \in \mathcal{C}^\infty(Q;F)$ may be written

$$s = \sum_{k=1}^{d_F} s_k(q) f_k \quad \text{with } \mathfrak{f} = \oplus_{k=1}^{d_F} \mathbb{C} f_k$$

and the covariant derivative $\nabla^F_{\partial_{q^j}} s$ equals

$$\nabla^F_{\partial_{q^j}}[s(q)] = \sum_{k=1}^{d_F} (\partial_{q^j} s_k)(q) f_k + s(q)(\nabla^F_{\partial_{q^j}} f_k)\,.$$

This connection is compatible with the hermitian structure of F when

$$\partial_{q^j} \langle f_k\,,\, f_{k'}\rangle_{g^F} = \langle \nabla^F_{\partial_{q^j}} f_k\,,\, f_{k'}\rangle_{g^F} + \langle f_k\,,\, \nabla^F_{\partial_{q^j}} f_{k'}\rangle_{g^F}\,,$$

for all $(j,k,k') \in \{1,\ldots,d\} \times \{1,\ldots,d_F\}^2$.
When $\pi : X = T^*Q \to Q$ is the natural projection, we shall work with the fiber bundle $\pi_{F_X} : F_X = \pi^* F \to X$ which also equals $T^*Q \otimes_Q F = X \otimes_Q F$. The tangent bundle TX is decomposed into $TX = (TX)^H \oplus (TX)^V$ with

$$(T_x X)^H = \sum_{j=1}^{d} \mathbb{R} e_i \sim T_q Q \quad \text{with} \quad e_j = \partial_{q^j} + \Gamma^\ell_{ij} p_\ell \partial_{p_i}$$

$$\text{and} \qquad (T_x X)^V = \sum_{j=1}^{d} \mathbb{R} \partial_{p_j} \sim T_q^* Q\,,$$

where $x = (q,p)$ and the Γ^ℓ_{ij}'s are the Christoffel symbol for the metric g on Q (see Remark 6.1 for the sign convention). Then the connection on F_X is given by

$$\nabla^{F_X}_{e_j} = \nabla^F_{\partial_{q^j}} \quad , \quad \nabla^{F_X}_{\partial_{p_j}} = 0\,. \tag{94}$$

For $1 \leq j \leq \dim Q$, the covariant derivative $\nabla^{F_X}_{e_j}(sf) = \nabla^{F_X}_{e_j}[s(q,p)f]$ then equals

$$\nabla^{F_X}_{e_j}(sf) = (e_j s) f + s \nabla^F_{\partial_{q^j}} f = (\partial_{q^j} s) f + (\Gamma^\ell_{ij} p_\ell \partial_{p_i} s) + s \nabla^F_{\partial_{q^j}} f$$

while

$$\nabla^{F_X}_{\partial_{p_j}}(sf) = (\partial_{p_j} s) f\,.$$

When ∇^F is compatible with the metric g^F, ∇^{F_X} is compatible with $g^{F_X} = \pi^* g^F$.

Definition 8.8. *Assume that (F, ∇^F, g^F) is a hermitian fiber bundle on Q with the connection ∇^F and the metric g^F. Let $(F_X, g^{F_X}, \nabla^{F_X})$ be the pull-back, $F_X = \pi^* F$, by the projection $\pi : X = T^*Q \to Q$ with $g^{F_X} = \pi^* g^F$ and ∇^{F_X} defined by (94). A geometric Kramers-Fokker-Planck operator is a differential operator on $\mathcal{C}^\infty(X; F^X)$ of the form*

$$P^F_{\pm,Q,g} + M(q,p,\partial_p) = P^F_{\pm,Q,g} + M^0_j(q,p)\nabla^{F_X}_{\partial_{p_j}} + M^1(q,p)\,,$$

where

$$\begin{aligned} & P^F_{\pm,Q,g} = \pm g^{ij}(q) p_i \nabla^{F_X}_{e_j} + \mathcal{O}_{Q,g}\,, \\ \text{with} \quad & \mathcal{O}_{Q,g} = \frac{-\Delta_p + |p|^2_q}{2}\,, \\ & e_j = \partial_{q^j} + \Gamma^\ell_{ij} p_\ell \partial_{p_i}\,, \end{aligned}$$

and where M^0_j, $M^1 \in \mathcal{C}^\infty(X; End(\pi^* F))$ satisfy

$$\|\partial^\alpha_p \partial^\beta_q M^0_j(q,p)\| \leq C_{\alpha,\beta} \langle p \rangle_q^{-|\alpha|}\,, \tag{95}$$

$$\text{and} \quad \|\partial^\alpha_p \partial^\beta_q M^1(q,p)\| \leq C_{\alpha,\beta} \langle p \rangle_q^{1-|\alpha|}\,, \tag{96}$$

for all multi-indices $(\alpha, \beta) \in \mathbb{N}^{2d}$.

The spaces $L^2(Q; \mathcal{H}^{s'})$, $s' \in \mathbb{R}$, and $H^s(\overline{Q}; \mathcal{H}^{s'})$ are defined locally as spaces of sections of F_X.
Let us specify the framework for boundary conditions. The fiber bundle $\pi_{F_{\partial X}} : F_{\partial X} = \pi^* F_{\partial Q} \to \partial X$ is the pull-back of $F_{\partial Q}$ by the projection $\pi\big|_{\partial X} : \partial X \to \partial Q$ and equals $\partial X \otimes_{\partial Q} F_{\partial Q}$. Traces will lie in $L^2(\partial X, |p_1| dq' dp; F_{\partial X})$. We assume that $F_{\partial Q}$ is endowed with an involution $\mathbf{j} \in \mathcal{C}^\infty(\partial Q; \mathrm{End}(F_{\partial Q}))$ such that for the metric g^F, $\mathbf{j}^* = \mathbf{j} = \mathbf{j}^{-1}$. The regularity of $\mathbf{j}$ ensures that for all $q \in \partial Q$, the fiber F_q can be decomposed into the orthogonal direct sum $F_q = \ker(\mathbf{j}(q) - \mathrm{Id}) \oplus^\perp \ker(\mathbf{j}(q) + \mathrm{Id})$ where both parts have constant dimensions. Equivalently there exists a unitary mapping $U \in C^\infty(\mathrm{End}(F_{\partial Q}; \partial Q \times \mathfrak{f}))$ such that $U(q)\mathbf{j}U(q)^* = j$ for a.e. $q \in \partial Q$, where $j = j^* = j^{-1} \in \mathrm{End}(\mathfrak{f})$ is constant.
The mapping which associates to $\gamma(q,p) \in L^2(\partial X, |p_1| dq' dp; F_{\partial X})$ the function $U(q)\gamma(q,p) \in L^2(\partial X, |p_1| dq' dp; \mathfrak{f})$ is a unitary isomorphism.

Definition 8.9. *Let (F, g^F) be endowed with a unitary involution $\mathbf{j}$ which belongs to $\mathcal{C}^\infty(\partial Q; End(F_{\partial Q}))$. For $\gamma \in L^2_{loc}(\partial X, |p_1|dq'dp; F_{\partial X})$ we define*

$$\gamma_{ev}(q, p_1, p') = \Pi_{ev}\gamma(q, p_1, p') = \frac{\gamma(q, p_1, p') + \mathbf{j}(q')\gamma(q, -p_1, p')}{2}$$
$$\text{and} \qquad \gamma_{odd}(q, p_1, p') = \Pi_{odd}\gamma(q, p_1, p') = \frac{\gamma(q, p_1, p') - \mathbf{j}(q')\gamma(q, -p_1, p')}{2} .$$

A bounded operator $\mathbf{A}$ on $L^2(\partial X, |p_1|dq'dp; F_{\partial X})$ is admissible if $U(q)\mathbf{A}U^*(q)$ on $L^2(\partial X, |p_1|dq'dp; \mathfrak{f})$ has the form $A(q, |p_q|)$ and fulfills the conditions (5)(6) *and either* (7) *or* (8).

Proposition 8.10. *Assume that $\overline{Q}$ is compact or a compact perturbation of the euclidean half-space $\overline{\mathbb{R}}^d_-$ and set $\overline{X} = T^*\overline{Q}$. In the second case, the fiber bundle (F, g^F, ∇^F) is assumed to coincide with $(Q\times\mathfrak{f}, g^F_0, 0)$, $\partial_q g^F_0 \equiv 0$, while the pair $(F_{\partial Q}, \mathbf{j})$ is trivial outside a compact domain of $\overline{\mathbb{R}^d_-}$*
Let the geometric Kramers-Fokker-Planck operator $P^F_{\pm,Q,g}+M(q, p, \partial_p)$ satisfy the conditions of Definition 8.8 and assume that the bounded operator $\mathbf{A}$ on $L^2(\partial X, |p_1|dq'dp; F_{\partial X})$ is admissible according to Definition 8.9.
There exists a constant $C > 0$ such that the operator $C + K^F_{\pm,\mathbf{A},g,M} = C + P^F_{\pm,Q,g} + M(q, p, \partial_p)$ defined with the domain

$$D(K^F_{\pm,\mathbf{A},g,M}) = \left\{ u \in L^1(Q; \mathcal{H}^1) \,, \quad \begin{array}{l} [P^F_{\pm,Q,g} + M(q, p, \partial_p)]u \in L^2(X, dqdp; F_X) \\ \gamma_{odd}u = \pm\, sign\,(p_1)\mathbf{A}\gamma_{ev}u \end{array} \right\}$$

is maximal accretive and satisfies the same subelliptic estimates as in Theorem 1.1 when $\mathbf{A} = 0$ and as Theorem 1.2 when $\mathbf{A} \neq 0$.
When $\overline{Q}$ is compact, $K^F_{\pm,\mathbf{A},g}$ has a compact resolvent and its spectrum is discrete.
The domain $D(K^F_{\pm,\mathbf{A},g;M}) = D(K^F_{\pm,\mathbf{A},g;0})$ does not depend on $M = M(q, p, \partial_p)$.
In particular when $A = 0$ and for any such M ,

$$D(K^F_{\pm,\mathbf{A},g;M}) \cap \left\{ u \in \mathcal{C}^\infty_0(\overline{X}; F_X) \,, \partial_{q^1}u = \mathcal{O}(|q^1|^\infty) \text{ near } \partial X \right\}$$

is dense in $D(K^F_{\pm,\mathbf{A},g;M})$ endowed with its graph norm.
If additionally the connection ∇^F is compatible with the metric g^F , then the following properties are true: The integration by part identity

$$\|u\|^2_{L^2(Q;\mathcal{H}^1)} + \mathrm{Re}\,\langle \gamma_{ev}u \,,\, \mathbf{A}\gamma_{ev}(u)\rangle_{L^2(\partial X, |p_1|dq'dp; F_{\partial X})} = \mathrm{Re}\,\langle u \,,\, (K^F_{\pm,\mathbf{A},g;0} - i\lambda)u\rangle$$

holds for all $\lambda \in \mathbb{R}$ *and all* $u \in D(K^F_{\pm,\mathbf{A},g;0})$ *.*
The adjoint of $K^F_{\pm,\mathbf{A},g;M} = K^F_{\pm,\mathbf{A},g;0} + M^0_j(q,p)\nabla^{F_X}_{\partial_{p_j}} + M^1(q,p)$ *equals*

$$K^F_{\mp,\mathbf{A}^*,g;M^*} = K^F_{\mp,\mathbf{A}^*,g;0} - \nabla^{F_X}_{\partial_{p_j}} \circ M^0_j(q,p)^* + M^1(q,p)^*\,.$$

When $U\mathbf{A}U^* = \mathbf{A}^*$ *, with* $Uu(q,p) = u(q,-p)$ *, and* $M(q,-p,-\partial_p)$ *equals the formal adjoint* $M(q,p,\partial_p)^*$ *,* $K^F_{\pm,\mathbf{A},g,M}$ *satisfies the PT-symmetry* (93) *and*

$$\sigma(K^F_{\pm,\mathbf{A},g;M}) = \overline{\sigma(K^F_{\mp,\mathbf{A}^*,g;M^*})} = \overline{\sigma(K^F_{\pm,\mathbf{A},g;M})}\,.$$

Remark 8.11. *All the consequences listed in Subsection 8.1 are valid.*

Proof. **a)** After introducing the suitable finite partition of unity, we can assume $F = Q \times \mathfrak{f}$ and the problem is reduced to the comparison

$$\begin{aligned}
&\|(\hat{U}(q)[P^F_{\pm,Q,g} + M(q,p,\partial_p)]\hat{U}(q)^* - P_{\pm,Q,g} \otimes \mathrm{Id}_{\mathfrak{f}})u\| \\
&\qquad\qquad \leq \|M(q,p,\partial_p)u\| + \|g^{ij}(q)p_i(\hat{U}(q)(\partial_{q^i}\hat{U}^*(q)) + \nabla^F_{\partial_{q^j}}u\| \\
&\qquad\qquad \leq C\|u\|_{L^2(Q;\mathcal{H}^1)}\,,
\end{aligned}$$

where the local unitary transform $\hat{U}(q)$: a) is nothing but Id when u in supported in an interior chart; b) trivializes $\mathbf{j}$ (and $\hat{U}\mathbf{A}\hat{U}^* = A(q,|p_q|)$) when the support of u meets ∂Q.
Consider the two realization $K^{g^F}_{\pm,A,g}$ and $K^{g^{\mathfrak{f}}}_{\pm,A,g}$ of $P_{\pm,Q,g} \otimes \mathrm{Id}$ when $Q \times \mathfrak{f}$ is endowed with the variable metric g^F and when it is endowed with the constant metric $g^{\mathfrak{f}}$. A simple conjugation shows that $K^{g^{\mathfrak{f}}}_{\pm,A,g}$ is unitarily equivalent to $K^{g^{\mathfrak{f}}}_{\pm,A,g} + M^1(q,p)$ where $M^1(q,p)$ satisfies (96). We have found a local unitary transform $\tilde{U}(q)$ from $L^2(X, dqdp; (\mathfrak{f}, g^{\mathfrak{f}}))$ to $L^2(X, dqdp; F_X)$ such that

$$\|P^F_{\pm,Q,g;M}u - \tilde{U}(q)^* K^{g^{\mathfrak{f}}}_{\pm,A,g}\tilde{U}(q)u\| \leq C\|u\|_{L^2(Q;\mathcal{H}^1)}\,,$$

for any $u \in D(K^F_{\pm,\mathbf{A},g})$ with $\mathrm{supp}\, u$ contained in a chart of the partition of unity. The subelliptic estimates hold for $K^{g^{\mathfrak{f}}}_{\pm,A,g}$ because it corresponds exactly to the situation of Theorem 1.1 and Theorem 1.2. Therefore $K^F_{\pm,,g}$ is locally a relatively bounded perturbation of $\tilde{U}^* K^{g^{\mathfrak{f}}}_{\pm,A,g}\tilde{U}$ and can be treated like the perturbation by a potential in Proposition 8.5, i.e. by applying the general perturbation result of Proposition 3.9. Putting together all the

pieces of the spatial partition of unity follows the line of Subsection 7.5. This proves the maximal accretivity of $C + K^F_{\pm,\mathbf{A},g;M}$, the extension of the subelliptic estimates of Theorem 1.1 and Theorem 1.2 and as well as the density statement when $A = 0$.
The global comparison

$$\forall u \in D(K_{\pm,\mathbf{A},g;0})\,, \quad \|(P^F_{\pm,Q,g} + M(q,p,\partial_p))u - K^F_{\pm,\mathbf{A},g;0}u\| \leq C\|u\|_{L^2(Q;\mathcal{H}^1)}\,,$$

implies that $K^F_{\pm,\mathbf{A},g;M}$ can be treated as a perturbation of $K^F_{\pm,\mathbf{A},g;0}$ like in Proposition 3.9 and $D(K^F_{\pm,\mathbf{A},g;M})$ does not depend on $M = M(q,p,\partial_p)$.
b) Assume now that the connection ∇^F is compatible with the metric g^F . Then the connection ∇^{F_X} is compatible with the metric g^{F_X} . After using a partition of unity in q and the integration by parts formula for the models $K^{g^{\mathfrak{f}}}_{\pm,A,g}$ for which all traces are well defined, one gets for all $u \in D(K^F_{\pm,A,g;0})$

$$\begin{aligned}
\mathbb{R}\mathrm{e}\,\langle u\,,\,K^F_{\pm,\mathbf{A},g;0}u\rangle &= \|u\|^2_{L^2(Q;\mathcal{H}^1)} \pm \int_X \mathcal{L}_{g^{ij}(q)p_ie_j}\left(\langle u\,,\,u\rangle_{g^F}\right)\,dqdp \\
&= \|u\|^2_{L^2(Q;\mathcal{H}^1)} \pm \int_{\partial X} \langle \gamma u\,,\,\mathrm{sign}\,(p_1)\gamma u\rangle_{g^F}\,|p_1|dq'dp \\
&= \|u\|^2_{L^2(Q;\mathcal{H}^1)} + \mathbb{R}\mathrm{e}\,\langle \gamma_{ev}u\,,\,A\gamma_{ev}u\rangle_{L^2(\partial X,|p_1|dq'dp;F)}\,.
\end{aligned}$$

The equality $(K^F_{\pm,A,g}) = K^F_{\mp,A^*,g}$ and the PT-symmetry when $UAU^* = A^*$ are deduced with the same argument from the compatibility of the connection with the hermitian structure. $\square$

9 Applications

In this section we introduce several kinds of boundary conditions and we check that they enter in our formalism. They are motivated by the stochastic process formulation or by the relationship between the geometric Kramers-Fokker-Planck equation (hypoelliptic Laplacian introduced by Bismut) and the Witten Laplacian. These boundary conditions will be introduced by doing formal calculations or by considering simple models. Complete justifications from the analysis or probabilistic point of view require additional work, for which our regularity results on the semigroup may be useful.

9.1 Scalar Kramers-Fokker-Planck equations in a domain of $\mathbb{R}^d$

Although the adjoint is obtained by changing the signed index $\pm$, we focus on the case of

$$P_{+,Q}(V) = p.\partial_q - \partial_q V(q).\partial_p + \frac{-\Delta_p + |p|^2}{2}\,. \qquad (97)$$

We recall without proofs standard results about the relationship between stochastic processes and parabolic PDE's on the euclidean space, $\mathbb{R}^d$. In particular we fix the interpretations in terms of (Kramers)-Fokker-Planck equations and Einstein-Smoluchowski equations. With this respect, it is simpler to discuss here the case when the force field is the gradient of a potential. This could be extended to the case of more general force fields while paying attention to the computation of adjoint operators. We refer the reader to [Ris][Nel] for a more detailed introduction and to [BisLNM] and [IkWa] for the extension to riemannian manifolds which is more involved. Then we introduce a way to think of boundary conditions as jump processes. Specific examples are discussed afterwards.

9.1.1 Einstein-Smoluchowski case

The pure spatial description of Brownian motion in a gradient field is provided by the stochastic differential equation

$$\left\{\begin{array}{l} dQ = -\nabla V(Q)dt + \sqrt{\frac{2}{\beta}}dW_t \\ Q_0 = q\,, \end{array}\right. \qquad (98)$$

where dW denotes the d-dimensional white noise with covariance matrix $E(dW(t)dW(t')) = \mathrm{Id}_{\mathbb{R}^d}\delta(t-t')$ while $\beta > 0$ is the inverse temperature. The function V is assumed to describe a confining $\mathcal{C}^\infty$ potential, the confinement being replaced by boundary conditions on a bounded domain. For $\tilde{u}_0 \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ the conditional expectation

$$\tilde{u}(q,t) = E(\tilde{u}_0(Q_t)|Q_0 = q)$$

solves the backward Kolmogorov equation

$$\left\{\begin{array}{l} \partial_t\tilde{u} = -\nabla V(q).\nabla_q\tilde{u} + \beta^{-1}\Delta_q\tilde{u} = -\beta^{-1}(-\partial_q + \beta\partial_q V(q)).\partial_q\tilde{u} = -\beta^{-1}L_{\beta V}\tilde{u} \\ \tilde{u}(q,0) = \tilde{u}_0(q)\,. \end{array}\right.$$

When Q_0 is randomly distributed with the law $\tilde{\mu}_0 = \tilde{\varrho}_0(q)dq$, the expectation $E(\tilde{u}_0(Q_t))$ equals

$$\begin{aligned} E(\tilde{u}_0(Q_t)) &= \int_{\mathbb{R}^d} \tilde{u}(q,t)\ d\tilde{\mu}_0(q) = \int_{\mathbb{R}^d} (e^{-t\beta^{-1}L_{\beta V}}\tilde{u}_0)\ d\tilde{\mu}_0 \\ &= \int_{\mathbb{R}^d} \tilde{u}_0(q)\ d\tilde{\mu}_t(q) = \int_{\mathbb{R}^d} \tilde{u}_0(q)\tilde{\varrho}(q,t)\ dq\,, \end{aligned}$$

where $\tilde{\mu}_t = e^{-t\beta^{-1}L^*_{\beta V}}\tilde{\mu}_0$, or the density $\tilde{\varrho}(t) = e^{-t\beta^{-1}L^*_{\beta V}}\tilde{\varrho}_0$, solves the Fokker-Planck equation

$$\begin{cases} \partial_t\tilde{\varrho} = \nabla_q.(\nabla V(q)\tilde{\varrho}) + \beta^{-1}\Delta_q\tilde{\varrho} = \beta^{-1}\partial_q.(\partial_q + \beta\partial_q V(q))\tilde{\varrho} = -\beta^{-1}L^*_{\beta V}\tilde{\varrho} \\ \tilde{\varrho}(q,0) = \tilde{\varrho}_0(q)\,. \end{cases}$$

With rather general confining assumptions (e.g. $|\partial_q V(q)| \geq C^{-1}|q|^\delta - C$ for $\delta > 0$, $\lim_{q\to\infty} V(q) = +\infty$ and $|\mathrm{Hess}\, V(q)| = o(|\nabla_q V(q)|)$ as $q \to \infty$) $\mu_\infty = \frac{e^{-\beta V(q)}dq}{\int_{\mathbb{R}^d} e^{-\beta V(q)}\ dq} = \varrho_\infty(q)\ dq$ is the unique invariant measure and the backward Kolmogorov equation can be studied in $L^p(\mathbb{R}^d; d\mu_\infty)$ for $p \in [1, +\infty]$. Especially when $p = 2$, setting $u = e^{-\frac{\beta V(q)}{2}}\tilde{u} \in L^2(\mathbb{R}^d, dq)$ the backward Kolmogorov equation is transformed into

$$\begin{cases} \partial_t u = -\Delta^{(0)}_{\beta V/2} u \\ u(q,0) = u_0(q) = e^{-\frac{\beta V(q)}{2}}\tilde{u}_0(q)\,, \end{cases}$$

where $\Delta^{(0)}_{\beta V/2}$ is the Witten Laplacian acting on functions

$$\Delta^{(0)}_{\beta V/2} = (-\partial_q + \frac{\beta}{2}\partial_q V(q)).(\partial_q + \frac{\beta}{2}\partial_q V(q)) = -\Delta_q + \frac{\beta^2}{4}|\nabla_q V(q)|^2 - \frac{\beta}{2}\Delta V(q)\,.$$

With the suitable confining assumptions, $\Delta^{(0)}_{\beta V/2}$ is self-adjoint on $L^2(\mathbb{R}^d, dq)$, its resolvent is compact and $\ker(\Delta^{(0)}_{\beta V/2}) = \mathbb{C}e^{-\frac{\beta V(q)}{2}}$.
By setting $\varrho(q,t) = e^{\frac{\beta V(q)}{2}}\tilde{\varrho}(q,t)$, the Fokker-Planck equation is also transformed into

$$\begin{cases} \partial_t \varrho = -\Delta^{(0)}_{\beta V/2}\varrho \\ \varrho(q,0) = \varrho_0(q) = e^{\frac{\beta V(q)}{2}}\tilde{\varrho}_0(q)\,. \end{cases}$$

This property and the self-adjointness of $\Delta^{(0)}_{\beta V/2}$ is summarized by saying that the process (98) is reversible. Note the relation

$$E(\tilde{u}_0(Q_t)) = \int_{\mathbb{R}^d} \tilde{u}(q,t)\tilde{\varrho}_0(q)\ dq = \int_{\mathbb{R}^d} u(q,t)\varrho_0(q)\ dq = \int_{\mathbb{R}^d} u_0(q)\varrho(q,t)\ dq\,,$$

when Q_0 is randomly distributed according to $\tilde{\varrho}_0(q)\ dq = e^{-\frac{\beta V(q)}{2}}\varrho_0(q)\ dq$.

9.1.2 Langevin stochastic process

The Langevin theory of Brownian motion takes place in the phase-space $x = (q,p) \in \mathbb{R}^{2d}$ and the stochastic differential equation is written

$$\begin{cases} dq = pdt \\ dp = -\partial_q V(q)dt - \nu pdt + \sqrt{\frac{2\nu}{\beta}}dW \end{cases} \tag{99}$$

where dW denotes again the d-dimensional white noise with covariance matrix $E(dW(t)dW(t')) = \mathrm{Id}_{\mathbb{R}^d}\delta(t-t')$, $\beta > 0$ is the inverse temperature and $\nu > 0$ is the friction coefficient (the mass is set to 1 here). It is a non reversible process which has nevertheless very strong relations with the reversible Einstein-Smoluchowski case. Actually the process (98) can be proven to be some weak limit, in the large friction regime $\nu \to +\infty$, of the Langevin process (99). We refer the reader to [Nel] for a simple approach. Within the L^2-framework, a more general approach proving the relationship between the hypoelliptic Laplacian introduced by Bismut and the Witten Laplacian has been considered in [BiLe], using a Schur complement technique in the spirit of [SjZw].
We shall use the notation $X_t = (q_t, p_t)$. For $\tilde{u}_0 \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$, the expectation

$$\tilde{u}(x,t) = E(\tilde{u}_0(X_t)|X_0 = x)\,,$$

solves

$$\begin{cases} \partial_t \tilde{u} = p.\partial_q \tilde{u} - \partial_q V(q).\partial_p \tilde{u} - \nu p.\partial_p \tilde{u} + \frac{\nu}{\beta}\Delta_p \tilde{u} = -B_V \tilde{u} \\ \tilde{u}(x,0) = \tilde{u}_0(x)\,. \end{cases}$$

When X_0 is randomly distributed with the law $\tilde{\mu}_0 = \tilde{\varrho}_0 dqdp$, the expectation $E(\tilde{u}_0(X))$ equals

$$\begin{aligned} E(\tilde{u}_0(X_t)) &= \int_{\mathbb{R}^{2d}} \tilde{u}(x,t)\ d\tilde{\mu}_0(x) = \int_{\mathbb{R}^d} (e^{-tB_V}\tilde{u}_0)\ d\tilde{\mu}_0 \\ &= \int_{\mathbb{R}^{2d}} \tilde{u}_0(x)\ d\tilde{\mu}_t(x) = \int_{\mathbb{R}^{2d}} \tilde{u}_0(x)\tilde{\varrho}(x,t)\ dx\,, \end{aligned}$$

where $\tilde{\mu}_t = e^{-tB_V^*}\tilde{\mu}_0$ or the density $\tilde{\varrho}(t) = e^{-tB_V^*}\tilde{\varrho}_0$, solves the equation

$$\begin{cases} \partial_t \tilde{\varrho} = -p.\partial_q \tilde{\varrho} + \partial_q V(q).\partial_p \tilde{\varrho} + \nu \partial_p.(p\tilde{\varrho}) + \frac{\nu}{\beta}\Delta_p \tilde{\varrho} = -\tilde{B}_V^* \tilde{\varrho} \\ \tilde{\varrho}(x,0) = \tilde{\varrho}_0(x)\,. \end{cases}$$

With a confining potential the unique invariant measure is the Maxwellian probability $\tilde{\mu}_\infty(q,p) = \frac{e^{-\beta(\frac{|p|^2}{2}+V(q))}dqdp}{\int_{\mathbb{R}^{2d}} e^{-\beta(\frac{|p|^2}{2}+V(q))}dqdp}$. As we did for the Einstein-Smoluchowski case, the PDE approach for the L^2-theory is better understood after taking

$$u(q,p,t) = e^{-\frac{\beta}{2}(\frac{|p|^2}{2}+V(q))}\tilde{u}(q,p,t) \quad \text{and} \quad \varrho(q,p,t) = e^{\frac{\beta}{2}(\frac{|p|^2}{2}+V(q))}\tilde{\varrho}(q,p,t)\,.$$

The conjugation relations

$$\begin{aligned} & e^{\pm\frac{\beta}{2}(\frac{|p|^2}{2}+V(q))}\partial_p e^{\mp\frac{\beta}{2}(\frac{|p|^2}{2}+V(q))} = \partial_p \mp \frac{\beta}{2}p \\ \text{and} \quad & e^{\pm\frac{\beta}{2}(\frac{|p|^2}{2}+V(q))}(p.\partial_q - \partial_q V(q).\partial_p)e^{\mp\frac{\beta}{2}(\frac{|p|^2}{2}+V(q))} = (p.\partial_q - \partial_q V(q).\partial_p)\,, \end{aligned}$$

imply that $\varrho(x,t)$ and $u(x,t)$ solve respectively

$$\left\{\begin{array}{l} \partial_t\varrho = -p.\partial_q\varrho + \partial_q V(q).\partial_p\varrho + \frac{\nu}{\beta}(\partial_p - \frac{\beta}{2}p)(\partial_p + \frac{\beta}{2}p)\varrho \\ \varrho(x,0) = \varrho_0(x) = e^{\frac{\beta}{2}(\frac{|p|^2}{2}+V(q))}\tilde{\varrho}_0(x)\,. \end{array}\right.$$

and

$$\left\{\begin{array}{l} \partial_t u = p.\partial_q u - \partial_q V(q).\partial_p u + \frac{\nu}{\beta}(\partial_p - \frac{\beta}{2}p)(\partial_p + \frac{\beta}{2}p)u \\ u(x,0) = u_0(x) = e^{-\frac{\beta}{2}(\frac{|p|^2}{2}+V(q))}\tilde{u}_0(x)\,. \end{array}\right.$$

Like in the Einstein-Smoluchowski case, we have the relation

$$\begin{aligned} E(\tilde{u}_0(X_t)) &= \int_{\mathbb{R}^{2d}} \tilde{u}(q,p,t)\tilde{\varrho}_0(q,p)\ dqdp \\ &= \int_{\mathbb{R}^{2d}} u(q,p,t)\varrho_0(q,p)\ dqdp = \int_{\mathbb{R}^{2d}} u_0(q,p)\varrho(q,p,t)\ dqdp \end{aligned}$$

when X_0 is randomly distributed according to $\tilde{\varrho}_0\ dqdp = e^{\frac{\beta}{2}(\frac{|p|^2}{2}+V(q))}\varrho_0(q,p)\ dqdp$. Taking $\beta = 2$ and $\nu = 1$ gives

$$\partial_t\varrho = -(P_{+,\mathbb{R}^d} - \frac{d}{2})\varrho \quad \text{and} \quad \partial_t u = -(P_{-,\mathbb{R}^d} - \frac{d}{2})u\,.$$

The operator (97) corresponds to the Kramers-Fokker-Planck equation in the sense that it provides the evolution of probability measures (divided by the square root of the Maxwellian density). In [HelNi] it was proved

that for any $V \in \mathcal{C}^\infty(\mathbb{R}^d)$, $P_{\pm,\mathbb{R}^d}$ initially defined on $\mathcal{C}_0^\infty(\mathbb{R}^{2d})$ has a unique maximal accretive extension $K_{\pm,\mathbb{R}^d}$. We can write $\varrho(t) = e^{-t(K_{+,\mathbb{R}^d}-\frac{d}{2})}\varrho_0$ and $u(t) = e^{-t(K_{-,\mathbb{R}^d}-\frac{d}{2})}u_0$. The process is no more reversible but up to the conjugation by the exponential factors, the adjoint evolution is obtained by changing p into $-p$ (see Subsection 8.2). This works only for gradient force fields.
Finally keep in mind that the choice $\beta = 2$ corresponds to our normalization of the Kramers-Fokker-Planck operator.

9.1.3 Jump process at the boundary

This paragraph is a heuristic introduction of boundary conditions for the Kramers-Fokker-Planck equation in terms of stochastic processes. We start with the Kramers-Fokker-Planck equation written for $\tilde{\varrho}(x,t) = e^{\frac{|p|^2}{2}+V(q)}\varrho(x,t)$ (remember $\beta = 2$) which is naturally interpreted as a probability density when there is no loss of mass. The domain Q is a regular bounded open domain of the euclidean space $\mathbb{R}^d$ and we set $\overline{Q} = Q \sqcup \partial Q$, $X = T^*Q$ and $\overline{X} = X \sqcup \partial X = T^*Q \sqcup T^*_{\partial Q}Q$. The Kramers-Fokker-Planck equation in the interior X reads

$$\left\{\begin{array}{l} \partial_t\tilde{\varrho} = -p.\partial_q\tilde{\varrho} + \partial_q V(q).\partial_p\tilde{\varrho} + \nu\partial_p.(p\tilde{\varrho}) + \frac{\nu}{\beta}\Delta_p\tilde{\varrho} \\ \tilde{\varrho}(x,0) = \tilde{\varrho}_0(x)\,, \end{array}\right. \tag{100}$$

and it has to be completed with boundary conditions along ∂X. We keep the conventions of the Introduction and take in a neighborhood $U \subset \overline{Q}$ of point $q_0 \in \partial Q$ a coordinate system such that $U \cap \partial Q = \{q^1 = 0\}$ and the euclidean metric on $\mathbb{R}^d$ follows (2):

$$|dq|^2 = (dq^1)^2 + m_{ij}(q^1, q')dq'^i dq'^j\,.$$

The corresponding symplectic coordinates on $T^*U \subset \overline{X}$ are written $(q,p) = (q^1, q', p_1, p')$, $p_1 > 0$ corresponding to the outgoing conormal component. The measure $|p_1|dq'dp$ on $\partial X = T^*_{\partial Q}Q$ does not depend on such a choice of coordinates and by introducing polar coordinates $p = |p|\omega$ with $\omega \in S^*_qQ \sim \mathbb{S}^{d-1}$, the same holds for $|\omega_1|dq'd\omega = |p|^{-d}|p_1|dq'dp$. The boundary ∂X is partitionned into

$$\partial X = \partial X^+ \sqcup \partial X^- \sqcup \partial X^0\,,$$

where ∂X^+ is the set of strictly outgoing rays

$$(0, q', p_1, p') \in \partial X \Leftrightarrow p_1 > 0\,,$$

∂X^- is the set of strictly incoming rays

$$(0,q',p_1,p') \in \partial X \Leftrightarrow p_1 < 0\,,$$

and ∂X^0 is the set of glancing rays

$$(0,q',p_1,p') \in \partial X \Leftrightarrow p_1 = 0\,.$$

By assuming that it makes sense, the trace of $\tilde{\varrho}$ along ∂X is denoted $\gamma\tilde{\varrho}$ and $\gamma_+\tilde{\varrho} = \gamma\tilde{\varrho}\big|_{\partial X^+}$ and $\gamma_-\tilde{\varrho} = \gamma\tilde{\varrho}\big|_{\partial X^-}$. When $\tilde{\varrho}(x,t)$ is smooth enough, a simple integration by parts gives

$$\frac{d}{dt}\int_X \tilde{\varrho}(q,p,t)~dqdp = \int_{\partial X^-} \gamma_-\tilde{\varrho}(q,p)~|p_1|dq'dp - \int_{\partial X^+}\gamma_+\tilde{\varrho}(q,p)~|p_1|dq'dp\,.$$

Here and in the sequel, the glancing region ∂X^0 is assumed to have a null measure contribution and the possible singularities around this region are absorbed by the weight $|p_1|$. The system is dissipative when the incoming flow $\int_{\partial X^-}\gamma_-\tilde{\varrho}(q,p)~|p_1|dq'dp$ is less than the outgoing flow $\int_{\partial X^+}\gamma_+\tilde{\varrho}(q,p)~|p_1|dq'dp$. The difference of fluxes may not vanish and it is convenient to send the missing mass to an artificial exterior point $\mathfrak{e}$. The relation between $\gamma_+\tilde{\varrho}|p_1|dq'dp$ and $\gamma_-\tilde{\varrho}|p_1|dq'dp$ can then be modelled by a Markov kernel $M(x^-,x^+)$ from ∂X^+ to $\partial X^-\sqcup\{\mathfrak{e}\}$: $M(x^-,x^+)$ is a $\partial X^-\sqcup\{\mathfrak{e}\}$-probability measure valued function of $x^+\in\partial X^+$ such that

$$\begin{aligned}\left[\int_{\partial X^+}\gamma_+\tilde{\varrho}(x^+)|p_1|dq'dp - \int_{\partial X^-}\gamma_-\tilde{\varrho}(x^-)|p_1|dq'dp\right]\delta_{\mathfrak{e}} + (\gamma_-\tilde{\varrho})(x^-)|p_1|dq'dp \\ = \int_{\partial X^+} M(x^-,x^+)\gamma_+\tilde{\varrho}(x^+)|p_1|dq'dp\,.\end{aligned}$$

While considering the stochastic process (99) with initial data $X_0\in X\sqcup\partial X^-$, the quantity

$$\tau_x = \inf\left\{t\in[0,+\infty)\,, X_t(x)\in\partial X^+\right\}\quad,\quad x\in X\sqcup\partial X^-\,,$$

is a stopping time. The Langevin stochastic process (99) can thus be completed as a set of cadlag trajectories in $\overline{X}\sqcup\{\mathfrak{e}\}$ by introducing a jump process J_M at times τ when $X_{\tau^-}\in\partial X^+$:

$$\begin{cases} dX_t = \begin{pmatrix} pdt \\ -\partial_q V(q)dt - pdt + dW\end{pmatrix} + dJ_M \quad \text{when } X_t = (q_t,p_t)\in\overline{X}\setminus\partial X^0 \\ dX_t = 0 \quad \text{for all } t\geq t_0 \text{ if } X_{t_0} = \partial X^0\sqcup\mathfrak{e}\,. \end{cases} \tag{101}$$

We chose the coefficients $\nu = 1$ and $\beta = 2$ according to the discussion of the previous paragraph. The transition matrix of the jump process J_M is given by the Markov kernel $M(x^-, x^+)$.
In order to be consistent with our framework, the jump process sending x^+ to x^- is assumed to be local with respect to $q \in \partial Q$, which means $x^- = (q, p^-)$ or $x^- = \mathfrak{e}$ when $x^+ = (q, p^+)$, and elastic, which means $|p_-|^2 = |p_+|^2$ when $x^- \neq \mathfrak{e}$. More precisely the Markov kernel is assumed to satisfy

$$M(x^-, x^+)\big|_{\partial X^- \times \partial X^+} = \delta(q^+ - q^-)\delta(|p^-| - |p^+|)\tilde{R}(q, r, \omega^-, \omega^+)$$

where we used $q^+ = q^- = q$ and $p^\pm = r\omega^\pm$ with $r = |p^+| = |p^-|$ and $\omega^\pm \in S^\pm = \left\{\omega \in S^{d-1}\,,\ \pm\omega_1 > 0\right\}$. Allowing $\omega^- \in S^- \sqcup \mathfrak{e}$, $\tilde{R}$ can be considered as a Markov kernel from S^+ to $S^- \sqcup \{\mathfrak{e}\}$ such that

$$\begin{aligned}\left[\int_{S^+} \gamma_+ \tilde{\varrho}(r\omega^+)|\omega_1^+|d\omega^+ - \int_{S^-} \gamma_- \tilde{\varrho}(r\omega^-)|\omega_1^-|d\omega^-\right] \delta_{\mathfrak{e}} + \gamma_- \tilde{\varrho}(r\omega^-)|\omega_1^-|d\omega^- \\ = \int_{S^+} \tilde{R}(q, r, \omega^-, \omega^+)\gamma_+ \tilde{\varrho}(q, r\omega^+)|\omega_1^+|d\omega^+\,,\end{aligned}$$

for all $(q, r) \in \partial Q \times (0, +\infty)$. For the regularity we assume first that for all $(q, r) \in \partial Q \times (0, +\infty)$, the Markov kernel sends $L^1(S^+, d\omega^+)$ into $L^1(S^- \sqcup \mathfrak{e}, d\omega^- \oplus \delta_{\mathfrak{e}})$.
Set

$$R(q, r, \omega^-, \omega^+) = |\omega_1^-|^{-1}\tilde{R}(q, r, \omega^-, \omega^+)|\omega_1^+| \quad \text{for } (\omega^-, \omega^+) \in S^- \times S^+\,,$$

and for $\gamma \in L^1(S^+, |\omega_1|d\omega)$, $\gamma' = R(q, r)\gamma$ when

$$\gamma'(\omega^-) = \int_{S^+} R(q, r, \omega^-, \omega^+)\gamma_+(\omega^+)\ d\omega^+ \quad \text{for } a.e.\ \omega^- \in S^-\,.$$

The two conditions

$$\tilde{R}(q, r, \{\mathfrak{e}\}, \omega^+) \geq \alpha \quad \text{for } a.e.\ \omega^+ \in S^+ \tag{102}$$
$$\text{and} \qquad R(q, p)1_{S^+} \leq 1_{S^-}\,, \tag{103}$$

imply for any $s \in [1, +\infty]$ (interpolate between $s = 1$ and $s = \infty$)

$$\forall \gamma \in L^s(S^+, |\omega_1|d\omega)\,, \quad \|R(q, p)\gamma\|_{L^s(S^-, |\omega_1|d\omega)} \leq (1 - \alpha)^{\frac{1}{s}}\|\gamma\|_{L^s(S^+, |\omega_1|d\omega)}\,. \tag{104}$$

When μ_t is the law of $X_t \in \overline{X} \sqcup \{\mathfrak{e}\}$ solving (101) with a suitable initial data μ_0, we claim that $\mu_t = \tilde{\varrho}(q,p,t)dqdp \oplus (1 - \int_X \tilde{\varrho}(q,p,t)~dqdp)\delta_{\mathfrak{e}}$ where $\tilde{\varrho}$ solves the Kramers-Fokker-Planck equation (100) with the boundary condition

$$\gamma_- \tilde{\varrho}(q, r.) = R(q,r)\gamma_+ \tilde{\varrho}(q, r.) \quad \text{for } a.e.\ (q,r) \in \partial Q \times (0, +\infty)\,,$$

shortly written $\gamma_- \tilde{\varrho} = R\gamma_+ \tilde{\varrho}$. Since the operator R preserves the energy (the jump process is local and elastic), replacing $\tilde{\varrho}$ by $\varrho(x,t) = e^{-(\frac{|p|^2}{2} + V(q))}\tilde{\varrho}(x,t)$ is straightforward:

$$\begin{cases} \partial_t \varrho = -P_{+,Q}\varrho\,, \\ \gamma_- \varrho = R\gamma_+ \varrho\,, \\ \varrho(x, t=0) = \varrho_0(x)\,. \end{cases} \tag{105}$$

Moreover the conditions (103) and (102) with a uniform lower bound $\alpha \geq 0$ imply that R is a contraction from $L^2(\partial X^+, |p_1|dq'dp)$ to $L^2(\partial X^-, |p_1|dq'dp)$ with norm $(1-\alpha)^{\frac{1}{2}}$. By following the arguments of Lemma 5.2 (with $\mathfrak{f} = \mathbb{C}$ and $j = 1$) the boundary condition can be rewritten $\Pi_{odd}\gamma\varrho = \mathrm{sign}\,(p_1)A\Pi_{ev}\gamma$ where A is a maximal accretive operator so that $\varrho(t) = e^{-tK_{+,A}}\varrho_0$. The identification of A with the corresponding assumptions will be specified in the examples listed below.

9.1.4 Comments

The first work to justify the stochastic trajectorial interpretation of boundary conditions for the Einstein-Smoluchowski equation and the Kramers-Fokker-Planck equation date back to Skorohod who introduced the so called Skorohod reflection map in [Sko1][Sko2]. The Einstein-Smoluchowski case, even when the force field is not a gradient field, has been widely studied because it relies on the standard regularity theory for elliptic boundary value problems. We refer the reader for example to [IkWa][StVa] and to [LiSz] where discontinuous boundary value problems and corner problems were also taken into account. For the Langevin process and the Kramers-Fokker-Planck equation little seems to be known. The weak formulation developed for kinetic theory in [Luc][Car] does not provide any information on the operator domain, nor any sufficient regularity. The one dimensional case has been studied with specular reflection in [Lap] and with some exotic non elastic case in [Ber]. More recently the half-space problem with specular reflection has been considered in [BoJa].

9.1.5 Specular reflection

The specular reflection is a deterministic jump process which transforms (p_1, p') into $(-p_1, p')$ when the particle hits the boundary $X_{\tau^-} \in \partial X^+$. Within the presentation of Paragraph 9.1.3 , the Markov kernel $\tilde{R}(q, r, \omega^-, \omega^+)$ is simply given by

$$\tilde{R}(q, r, \omega^-, \omega^+) = \delta(\omega_1^- + \omega_1^+)\delta(\omega'^- - \omega'^+)\,.$$

The stochastic process (101) takes place in $\overline{X}$ and all the mass of the probability measure $\tilde{\varrho}(q,p,t)dqdp = e^{-(\frac{|p|^2}{2}+V(q))}\varrho(q,p,t)dqdp$ lies in $\overline{X}$. We can forget the exterior $\mathfrak{e}$. The corresponding boundary condition for $\varrho(x,t)$ will be

$$\varrho(0, q', p_1, p) = \varrho(0, q', -p_1, p') \quad \text{for } p_1 < 0\,.$$

Within our formalism for boundary value problem for the Kramers-Fokker-Planck operator $P_{+,Q}$, it can be written

$$\begin{aligned} &\gamma_{odd} u(x,t) = 0 \\ \text{with} \quad &\mathfrak{f} = \mathbb{C} \quad \text{and} \quad j = 1\,. \end{aligned}$$

We can apply Theorem 1.1 (case $A = 0$) and its Corollaries of Subsection 8.1 (see Subsection 8.3 for adding a potential). The PT-symmetry (93) is also satisfied and Proposition 8.4 also applies.

The kernel $\mathrm{Ker}\,(K_{+,0})$ equals $\mathbb{C}\varrho_\infty$ with $\varrho_\infty(q,p) = e^{-(\frac{|p|^2}{2}+V(q))}$ and $\tilde{\mu}_\infty = \frac{e^{-2(\frac{|p|^2}{2}+V(q))}dqdp}{\int_{\overline{X}} e^{-2(\frac{|p|^2}{2}+V(q))}dqdp}$ is the equilibrium probability measure. Corollary 8.2 ensures the exponential return to the equilibrium in the L^2-sense for the Langevin process with specular reflection in a bounded domain.

We expect that in the large friction limit (introduce the parameter ν and consider the limit as $\nu \to +\infty$ while $\beta = 2$), the solution $\varrho(q,p,t) = e^{-tK_{+,0}}\varrho_0$ converges to $f(q,t)e^{-\frac{|p|^2}{2}}$ with $f(q,t) = e^{-t\Delta_V^{(0),N}} f_0$, where $\Delta_V^{(0),N}$ is the Neumann realization of the Witten Laplacian $\Delta_V^{(0)} = -\Delta + |\nabla V(q)|^2 - \Delta V(q)$. This realization is characterized by

$$u \in D(\Delta_V^{(0),N}) \Leftrightarrow \left\{ \begin{array}{ccc} u \in L^2(Q, dq) & , & \Delta_V^{(0)} u \in L^2(Q, dq)\,, \\ & & \partial_n u + \partial_n V(q) u\big|_{\partial Q} = 0\,. \end{array} \right\}$$

where ∂_n is the outgoing normal derivative. Intuitively it corresponds to putting a $+\infty$ repulsive potential outside $\overline{Q}$ and to the specular reflection within the phase-space dynamical picture.

9.1.6 Full absorption

Within the Einstein-Smoluchowski approach another natural boundary condition is the homogeneous Dirichlet boundary condition for $\tilde{\varrho}(q,t)$ or $\varrho(q,t)$. This means that the particles are absorbed by the exterior as soon as they hit the boundary ∂Q and it can be intuitively presented by putting a $-\infty$ attractive potential outside $\overline{Q}$. The evolution of $\varrho(q,t)$ is given by $\varrho(t) = e^{-t\Delta_V^{(0),D}}\varrho_0$ where $\Delta_V^{(0),D}$ is the Dirichlet realization of the Witten Laplacian $\Delta_V^{(0)} = -\Delta + |\nabla V(q)|^2 - \Delta V(q)$. Within the Langevin description this can be modelled by sending the particle to the exterior $\mathfrak{e}$ as soon as it hits ∂X^+. Within the presentation of Paragraph 9.1.3, this deterministic jump process is simply given by the Markov kernel

$$\tilde{R}(q,r,\omega^-,\omega^+) = \delta_{\mathfrak{e}}\,.$$

The corresponding boundary condition in (105) is simply

$$\varrho(0,q',p_1,p') = 0 \quad \text{for } p_1 < 0\,.$$

With our notations they can be simply written as

$$\begin{aligned} &\gamma_{odd}u(q,p) = \mathrm{sign}\,(p_1)\gamma_{ev}u(q,p) \quad , \quad q \in \partial Q\,, \\ \text{with} \quad &\mathfrak{f} = \mathbb{C} \quad \text{and} \quad j = 1\,. \end{aligned}$$

We are in the case when $A = 1$ and Theorem 1.2 and its Corollaries of Subsection 8.1 apply (see Subsection 8.3 for the case with the potential V). Since $j1j = 1$, the PT-symmetry (93) is fulfilled and Proposition 8.4 is valid.
We also expect some kind of convergence to the Einstein-Smoluchowski equation, with the generator $\Delta_V^{(0),D}$ after the conjugation with the exponential weight, in the large friction limit $\nu \to \infty$. Note nevertheless that the density $\int_{T_q^*Q}\varrho(q,p,t)\;dp$ does not vanish in general when $q \in \partial Q$. Some numerical simulations provided by T. Lelièvre in the one-dimensional case show that in the large friction limit exponentially decaying boundary layers persist in the neighborhood of ∂Q, while comparing the local quantity $\int_{\mathbb{R}^d} e^{-(\frac{|p|^2}{2}+V(q))}\varrho(q,p,t)\;dp$ and the solution to the Einstein-Smoluchowski equation with Dirichlet boundary conditions.
A reason for studying such boundary conditions is that they are related with quasi-stationnary distributions. We refer to [LBLLP] for the presentation for

the Einstein-Smoluchowski equation. A detailed analysis in the low temperature limit has been performed recently with the help of semiclassical tools for Witten Laplacians, with T. Lelièvre in [LeNi]. The Langevin phase-space approach is a more natural framework for molecular dynamics.

9.1.7 More general boundary conditions

In [Car] the weak formulation of the Kramers-Fokker-Planck equation is studied for more general conditions $\gamma_{-}\varrho(q,r.) = R(q,r)\gamma_{+}\varrho(q,r.)$ similar to (100). Within the assumptions introduced in Paragraph 9.1.3, let us check that they enter in our formalism. We assume in particular that the lower bound (102) holds with $\alpha > 0$ uniform w.r.t to $(q,r) \in \partial Q \times (0,+\infty)$. We shall work here with $\mathfrak{f} = \mathbb{C}$ and $j = 1$ so that the projections Π_{+} and Π_{-} defined on $L^2(\partial X, |p_1|dq'dp; \mathbb{C})$ (see Definition 4.1) are characterized by

$$\Pi_{+}\gamma(q,p_1,p') = \gamma(q,|p_1|,p') \quad , \quad \Pi_{-}\gamma(q,p_1,p') = \gamma(q,-|p_1|,p')\,.$$

The relation $\gamma_{-}\varrho(q,r.) = R(q,r)\gamma_{+}\varrho(q,r.)$ can be written $\Pi_{-}\gamma\varrho(q,r.) = R'(q,r)\Pi_{+}\gamma\varrho(q,r.)$ with

$$R'(q,r,\omega^*,\omega) = 1_{\mathbb{R}_-}(\omega_1^*)R(q,r,\omega^*,\omega)1_{\mathbb{R}_+}(\omega_1) + 1_{\mathbb{R}_+}(\omega_1^*)R(q,r,\widehat{\omega^*},\widehat{\omega})1_{\mathbb{R}_-}(\omega_1)\,,$$

with $\widehat{\omega} = (-\omega_1,\omega')$ when $\omega = (\omega_1,\omega')$. The estimate (104) imply that R' is a contraction of $\Pi_{ev}L^2(\partial X, |p_1|dq'dp)$ with the norm $(1-\alpha)^{\frac{1}{2}}$. The operator R' commutes with Π_{ev} and it is local in the variables $(q, r = |p|)$. Since R' is a strict contraction with norm $(1-\alpha)^{1/2} < 1$ we can define the bounded operator

$$A = \frac{1-R'}{1+R'}$$

which is local in $(q, r = |p|)$ and bounded in $L^2(\partial X, |p_1|dq'dp)$ with the norm $\|A\| \leq \frac{1+(1-\alpha)^{1/2}}{1-(1-\alpha)^{1/2}}$. Moreover the inequality

$$\begin{aligned}\|(1-A)u\|^2_{L^2(\partial X,|p_1|dq'dp)} &= \|R'(1+A)u\|^2_{L^2(\partial X,|p_1|dq'dp)}\\ &\leq (1-\alpha)\|(1+A)u\|^2_{L^2(\partial X,|p_1|dq'dp)}\end{aligned}$$

implies

$$\forall u \in L^2(\partial X, |p_1|dq'dp)\,, \quad \alpha\|u\|^2_{L^2(\partial X,|p_1|dq'dp)} \leq 4\,\mathbb{R}\mathrm{e}\,\langle u\,,\,Au\rangle_{L^2(\partial X,|p_1|dq'dp)}\,.$$

Hence the operator $A = A(q,r)$ fulfills all the conditions (5)(6)(7)(8). We can apply Theorem 1.2 and the corollaries of Subsection 8.1 and Subsection 8.3. Note in particular that $A = jAj = A^*$ ($j = 1$), when $R'(q,r,\omega^*,\omega)$ is symmetric in (ω^*,ω).
A specific case is when a particle hitting ∂X^+ at $x = (q, p_1, p')$, $p_1 > 0$, jumps to $(q, -p_1, p)$ with probability $\varepsilon(q,p)$ and to $\mathfrak{c}$ with probability $1 - \varepsilon(q,p)$. The Markov kernel is then

$$R(q,r,\omega^-,\omega^+) = \varepsilon(q,p)\delta(\omega_1^- + \omega_1^+)\delta(\omega'^- - \omega'^+) + (1-\varepsilon(q,p))\delta_{\mathfrak{c}}\,.$$

We assume $1 - \varepsilon(q,p) \geq \alpha > 0$ for some $\alpha > 0$ independent of $(q,p) \in \partial X^+$. Then the boundary condition reads simply

$$\gamma_-\varrho(q,-p_1,p') = \varepsilon(q,p_1,p')\gamma_+\varrho(q,p_1,p') \quad \text{with } 0 \leq \varepsilon(q,p) \leq 1-\alpha\,, \tag{106}$$

or

$$\gamma_{odd}\varrho(q,p) = \mathrm{sign}\,(p_1)\frac{1-\varepsilon(q,|p_1|,p')}{1+\varepsilon(q,|p_1|,p')}\gamma_{ev}\varrho(q,p) \quad \forall (q,p) \in \partial X\,.$$

Remark 9.1. *For boundary conditions of the form* (106)*, Theorem 1.1 applies to the case $\varepsilon \equiv 1$ which corresponds to specular reflection (see Paragraph 9.1.5) while Theorem 1.2 applies to the case* (106) *with the uniform upper bound $\varepsilon(q,p) \leq 1-\alpha$ with $\alpha > 0$. We are unfortunately not able to treat the general case when $\varepsilon(q,p) \leq 1$. A norm smaller than 1 for the contraction $C_\pm^L(\lambda)$ in Proposition 4.15 would suffice to handle the more general case $\varepsilon(q,p) \leq 1$. A result that we are not able to achieve in the general abstract setting of Section 4. Whether taking for $L_\pm$ a tangential Kramers-Fokker-Planck operator would help, is an open question.*
Another type of boundary conditions which occur sometimes in kinetic theory and that we are not able to treat with Theorem 1.1 and Theorem 1.2 is the case of bounce-back boundary conditions. It corresponds to the case when a particle hitting the boundary ∂X^+ is sent back with an opposite velocity:

$$\gamma_-\varrho(q,-p) = \gamma_+\varrho(q,p) \quad \textit{when } (q,p) \in \partial X^+\,.$$

The Markov kernel is then

$$R(q,r,\omega^-,\omega^+) = \delta(\omega^- + \omega^+)$$

and it does not lead to a strict contraction in $\Pi_{ev}L^2(\partial X, |p_1|dq'dp)$.

9.1.8 Change of sign

Forget for a while the potential V. Take $V = 0$ so that the Witten Laplacian $\Delta_V^{(0)}$ is the opposite standard Laplacian $-\Delta$ and consider a half-space problem $Q = \mathbb{R}^d_- = \{(q^1, q'), q^1 < 0\}$. In Paragraph 9.1.5 (resp. Paragraph 9.1.6) the Neumann (resp. Dirichlet) boundary conditions for $-\Delta$ where interpreted as a reflecting (resp. absorbing) boundary ∂Q for the stochastic process and this led us to natural boundary conditions for the Kramers-Fokker-Planck equation. Here is another phase-space version of the Dirichlet case. By embedding the half-space problem into a whole-space problem with the symmetry $q^1 \to -q^1$, the Neumann (resp. Dirichlet) boundary conditions can be introduced by considering even (resp. odd) elements of $L^2(\mathbb{R}^d, dq)$. More precisely the boundary value problem

$$(1-\Delta)u = f \quad \text{with } \partial_{q^1} u\big|_{\partial Q} = 0 \text{ (resp. } u\big|_{\partial Q} = 0) ,$$

for $f \in L^2(Q, dq)$ is equivalent to

$$(1-\Delta)\tilde{u} = \tilde{f}$$

where $\tilde{u}, \tilde{f} \in L^2(\mathbb{R}^d, dq)$ are the even (resp. odd) extensions of u and f.
In the phase-space $\mathbb{R}^{2d}$ with $X = T^*Q = \mathbb{R}^{2d}_- = \{(q^1, q', p_1, p'), q^1 < 0\}$, the symmetry which preserves the Kramers-Fokker-Planck operator is

$$(q^1, q', p_1, p') \to (-q^1, p', -p_1, p')$$

and the even (resp. odd) extension of $u \in L^2(\mathbb{R}^{2d}_-, dqdp)$ is given by the operator Σ of Definition 5.3

$$\Sigma u(q^1, q', p_1, p') = \begin{cases} u(q^1, q', p_1, p') & \text{if } q^1 < 0 , \\ ju(-q^1, q', -p_1, p') & \text{if } q^1 > 0 , \end{cases}$$

with $j = 1$ (resp. $j = -1$). A solution to $P_{+,Q}u = f$ with the specular reflection boundary condition is equivalent to $P_{+,\mathbb{R}^d}(\Sigma u) = \Sigma f$ with $j = 1$. When we take $j = -1$ the boundary condition

$$\gamma_{odd}\varrho = 0$$

is equivalent to

$$\gamma_-\varrho(q, -p_1, p') = -\gamma_+(q, p_1, p') \quad , \quad \forall (q,p) \in \partial X^+ . \tag{107}$$

Now when $Q \subset \mathbb{R}^d$ is a bounded regular domain of $\mathbb{R}^d$, Theorem 1.1 with $\mathfrak{f} = \mathbb{C}$ and $j = -1$ applies, as well at its corollaries of Subsection 8.1 and Subsection 8.2. Adding a potential V is treated in Subsection 8.3.
We expect again that in the large friction limit (add the parameter $\nu > 0$ and let $\nu \to \infty$), the solution of the Kramers-Fokker-Planck equation converges to $f(q,t)e^{-\frac{|p|^2}{2}}$ with $f(t) = e^{-t\Delta_V^{(0),D}} f_0$. The good point of the boundary condition (107) is that the density $\int_{T_q^*Q} \varrho(q,p)\ dp$ vanishes for all $q \in \partial Q$. The drawback is that the Kramers-Fokker-Planck equation with such boundary conditions does not preserve the positivity. There is no straightforward interpretation in terms of stochastic processes.

9.2 Hypoelliptic Laplacian

In a series of works J.M. Bismut (see for example [Bis05][Bis1][BiLe]) introduced and analyzed what he called the hypoelliptic Laplacian. It is the phase-space version of Witten's deformation of Hodge theory and it leads to an hypoelliptic non self-adjoint second order operator. The main part of it is actually the scalar Kramers-Fokker-Planck operator. As there are natural boundary conditions for Witten or Hodge Laplacians, which extends the scalar Dirichlet an Neumann boundary conditions to p-forms and correspond respectively to the relative homology and absolute homology (see [ChLi][HeNi1][Lau][Lep3][LNV]), we propose a phase-space version for the hypoelliptic Laplacian. After recalling the writing of the Witten Laplacian and of the hypoelliptic Laplacian, a simple half-space problem will first be considered with symmetry arguments like in Paragraph 9.1.8. We finally propose a general version of these boundary conditions and check that they enter in our formalism. Especially in this section, we will see the interest of a fiber bundle presentation of Subsection 8.4 with fiber $\mathfrak{f}$ and a general involution j (or $\mathbf{j}$).

9.2.1 Witten Laplacian and Bismut's hypoelliptic Laplacian

We consider here the case when $\overline{Q} = Q$ is a riemannian manifold without boundary. The exterior fiber bundle is denoted by $\bigwedge T^*Q = \oplus_{p=0}^d \bigwedge^p T^*Q$ and we shall consider its flat complexified version $(\bigwedge T^*Q) \otimes_Q (Q \times \mathbb{C}) = \sqcup_{q\in Q}(\bigwedge T_q^*Q) \otimes \mathbb{C}$. We shall use the shorter notation $\bigwedge T^*Q \otimes \mathbb{C}$ when there is no ambiguity. The set of $\mathcal{C}_0^\infty$ differential forms is $\mathcal{C}_0^\infty(Q; \bigwedge T^*Q \otimes \mathbb{C}) = \bigoplus_{p=0}^d \mathcal{C}_0^\infty(Q; \bigwedge^p T^*Q \otimes_Q (Q \times \mathbb{C}))$ (with $\mathcal{C}_0^\infty = \mathcal{C}^\infty$ when $\overline{Q} = Q$ is compact).

The metric is denoted by $g = g_{ij}(q)dq^i dq^j$ and it provides a natural Hermitian bundle structure on $\bigwedge T^*Q \otimes \mathbb{C}$. When $d\mathrm{Vol}_g(q)$ is the riemannian volume, the L^2-scalar product of two differential forms equals

$$\langle \omega, \eta \rangle = \int_Q \langle \omega(q)\, ,\, \eta(q) \rangle_{g(q)}\ d\mathrm{Vol}_g(q)\, .$$

The differential acting on $\mathcal{C}_0^\infty(Q; \bigwedge T^*Q \otimes \mathbb{C}))$ is denoted by d: For $\omega = \sum_{\sharp I = p} \omega_I(q) dq^I \in \mathcal{C}_0^\infty(Q; \bigwedge^p T^*Q \otimes \mathbb{C})$

$$d\left(\sum_{\sharp I = p} \omega_I(q) dq^I\right) = \sum_{i=1}^d \sum_{\sharp I = p} \partial_{q^i} \omega_I(q)\ dq^i \wedge dq^I\ \ \in \mathcal{C}_0^\infty(Q; \bigwedge^{p+1} T^*Q \otimes \mathbb{C})\, .$$

The codifferential d^* is its formal adjoint for the above L^2-scalar product. For $V \in \mathcal{C}^\infty(Q; \mathbb{R})$, Witten's deformations of the differential and codifferential (see [CFKS][Wit][Zha]) are respectively given by

$$d_V = e^{-V} d e^V = d + dV \wedge \quad , \quad d_V^* = e^V d^* e^{-V} = d^* + \mathbf{i}_{\nabla V}\, .$$

Owing to $d \circ d = 0$, they also satisfy

$$d_V \circ d_V = 0 \quad , \quad d_V^* \circ d_V^* = 0\, .$$

The Witten Laplacian equals

$$\Delta_V = (d_V + d_V^*)^2 = d_V^* d_V + d_V d_V^* = \bigoplus_{p=0}^d \Delta_V^{(p)}$$

with

$$\Delta_V^{(0)} = -\Delta_q + |\nabla V(q)|^2 - \Delta_V(q)$$

and more generally

$$\Delta_V = (d + d^*)^2 + |\nabla V(q)|^2 + \mathcal{L}_{\nabla V} + \mathcal{L}_{\nabla V(q)}^*\, ,$$

where $\mathcal{L}_X$ is the Lie derivative along the vector field X. When $V = 0$, the Witten Laplacian is nothing but the Hodge Laplacian. An important property, which requires a specific attention when $\partial Q \neq \emptyset$ and boundary conditions are added, is

$$\Delta_V \circ d_V = d_V \circ \Delta_V \quad , \quad \Delta_V \circ d_V^* = d_V^* \circ \Delta_V\, .$$

Bismut's hypoelliptic Laplacian is constructed like Witten's Laplacian, now on the phase-space $X = T^*Q$. The differential d^X is defined as usual on $\mathcal{C}_0^\infty(X; \bigwedge T^*X \otimes \mathbb{C})$, with $\bigwedge T^*X \otimes \mathbb{C} = (\bigwedge T^*X) \otimes_X (X \times \mathbb{C})$. But the codifferential $\overline{d}^X_{\phi_b}$ is now defined with respect to a non degenerate but non hermitian sesquilinear form and the weight $e^{V(q)}$ has to be replaced by $e^{\mathcal{H}(x)} = e^{\mathcal{H}(q,p)}$, where $\mathcal{H}(q,p)$ is some energy functional on the phase-space. The sesquilinear form is given by

$$\langle s\,,\, s'\rangle_{\phi_b} = \int_X \langle s(x)\,,\, s'(x)\rangle_{\eta_b^*}\; dqdp\,,$$

where $dqdp$ is the symplectic volume on X and η_b^* is the dual form, extended to $\bigwedge T^*X$ and a trivial hermitian version on $\bigwedge T^*X \otimes \mathbb{C}$, of

$$\eta_b(U,V) = \langle \pi_* U\,,\, \pi_* V\rangle_g + b\omega(U,V)\,, \quad U,V \in TX = T(T^*Q)\,.$$

In the above definition $\pi : X = T^*Q \to Q$ is the natural projection and $\omega = dp_j \wedge dq^j$ is the symplectic form (an element $p \in T^*_q Q$ is written $p = p_j dq^j$). The mapping $\phi_b : TX \to T^*X$ is given by $\eta_b(U,V) = \langle U\,,\, \phi_b V\rangle$. For a section $s \in \mathcal{C}^\infty(X; \bigwedge T^*X \otimes \mathbb{C})$, $\overline{d}^X_{\phi_b} s$ is then defined by

$$\forall s' \in \mathcal{C}_0^\infty(X; T^*X \otimes \mathbb{C})\,, \quad \langle s\,,\, d^X s'\rangle_{\phi_b} = \langle \overline{d}^X_{\phi_b} s\,,\, s'\rangle_{\phi_b}\,.$$

The function $\mathcal{H}(q,p) = \frac{1}{2}|p|_q^2 + V(q) = \mathcal{E}(q,p) + V(q)$ and the corresponding Hamiltonian vector field on $X = T^*Q$ endowed with the symplectic form ω is

$$\mathcal{Y}_{\mathcal{H}} = \mathcal{Y}_{\mathcal{E}} + \mathcal{Y}_V = g^{ij}(q) p_i e_j - \partial_{q^j} V(q) \partial_{p_j}\,,$$

after introducing the vector field $e_j = \partial_{q^j} + \Gamma^\ell_{kj} p_\ell \partial_{p_k}$ (see Subsection 6.1 and Remark 6.1).
The deformed differential and codifferential are then given by

$$d^X_{\mathcal{H}} = e^{-\mathcal{H}} d^X e^{\mathcal{H}} \quad \text{and} \quad \overline{d}^X_{\phi_b,\mathcal{H}} = e^{\mathcal{H}} \overline{d}^X_{\phi_b} e^{-\mathcal{H}}\,,$$

and the hypoelliptic Laplacian by the square

$$\mathcal{U}^2_{\phi_b,\mathcal{H}} = \frac{1}{4}(\overline{d}^X_{\phi_b,\mathcal{H}} + d^X_{\mathcal{H}})^2 = \frac{1}{4}(\overline{d}^X_{\phi_b,\mathcal{H}} \circ d^X_{\mathcal{H}} + d^X_{\mathcal{H}} \circ \overline{d}^X_{\phi_b,\mathcal{H}})\,. \tag{108}$$

The Weitzenbock type formula of [Bis05] expresses $\mathcal{U}^2_{\phi_b,\mathcal{H}}$ in a coordinate system or as a sum of elementary geometric operators. It relies on the

identification $T_xT^*X \sim T_qQ \oplus T_q^*Q$ related with the adjoint Levi-Civita connection on $X = T^*Q$ associated with the riemannian metric g. For $x = (q,p) \in X = T^*Q$, the adjoint Levi-Civita connection on T^*Q provides the vertical-horizontal decomposition $T_xX = (T_xX)^H \oplus (T_xX)^V$ where $(T_xX)^V \sim T_q^*Q$ is spanned by $(\hat{e}^j = \partial_{p_j})_{j=1,\ldots,d}$ and $(T_xX)^H \sim T_qQ$ is spanned by $(e_j = \partial_{q^j} + \Gamma_{ij}^{\ell}p_\ell\partial_{p_i})$. The horizontal tangent vector $\partial_{q^j} + \Gamma_{ij}^{\ell}p_\ell\partial_{p_i}$ is sent to ∂_{q^j} by the isomorphism $\pi_*\big|_{(T_xX)^H} : (T_xX)^H \to T_qQ$. The dual basis of $(e_j, \hat{e}^j)_{j=1,\ldots,d}$ is denoted by $(e^j, \hat{e}_j)_{j=1,\ldots,d}$ with $e^j = dq^j$ and $\hat{e}_j = dp_j - \Gamma_{ij}^{\ell}p_\ell dq^j$. Note that $(e^j)_{j=1,\ldots,d}$ (resp. $(\hat{e}_j)_{j=1,\ldots,d}$) is nothing but another copy of $(\hat{e}^j)_{j=1,\ldots,d}$ (resp. of $(e_j)_{j=1,\ldots,d}$) after identifying $(T_x^*X)^H$ with T_q^*Q (resp. $(T_x^*X)^V$ with T_qQ). Hence $\bigwedge^1 T_x^*X \sim \bigwedge^1 T_q^*Q \oplus \bigwedge^1 T_qQ$ and the exterior algebra $\bigwedge T_x^*X$ is identified with $(\bigwedge T_q^*Q) \otimes (\bigwedge T_qQ)$. Some details are given below.
According to Theorem 3.7 of [Bis05] with $\beta = \frac{1}{b}$ (see also Theorem 3.8 of [Bis05] and Theorem 3.7 in [Bis1] for some simplifications), the hypoelliptic Laplacian equals

$$\begin{aligned}\mathcal{U}_{b,\mathcal{H}}^2 =& \frac{1}{4b^2}\big[-\Delta_p + |p|^2 + 2(\hat{e}_i\wedge)\mathbf{i}_{\hat{e}^i} - d \\ &\qquad - \frac{1}{2}\langle R^{TX}(e_i,e_j)e_k\,,\, e_\ell\rangle(e^i\wedge)(e^j\wedge)\mathbf{i}_{\hat{e}^k}\mathbf{i}_{\hat{e}^\ell}\big] - \frac{1}{2b}\mathcal{L}_{\mathcal{Y}^{\mathcal{H}}}\,, \end{aligned} \tag{109}$$

where R^{TX} is the Riemann curvature tensor on TQ.

Remark 9.2. *More precisely the Weitzenbock formula* (109) *is really written in this way in Theorem 3.7 and Theorem 3.8 of [Bis05] for $V \equiv 0$ (take $\omega(F, \nabla F) = 0$ with the notations of [Bis05]). The way to include a non zero potential $V(q)$ is explained in Remark 2.37 of [Bis05]: It suffices to endow the complexification bundle $F_1 = Q \times \mathbb{C}$ with the metric $g^{F_1} = e^{-2V(q)}$ then to apply the Weitzenbock formula of Theorem 3.7 and Theorem 3.8 in [Bis05] with $\omega(\nabla^{F_1}, g^{F_1})$ (denoted by $\omega(\nabla^F, g^F)$ in [Bis05]) equal to $-2\nabla_q V(q)$ and finally to compute $e^{-V(q)}\mathcal{U}_{b,\mathcal{E}}^2 e^{V(q)}$ from the general formula for $\mathcal{U}_{b,\mathcal{E}}$.*

The hypoelliptic Laplacian $\mathcal{U}_{b,\mathcal{H}}$ is not yet in the form of a geometric Kramers-Fokker-Planck operator (see Definition 8.8). For this we need some precisions about the identification $\bigwedge T^*X \sim (\bigwedge T^*Q) \otimes_Q (\bigwedge TQ)$. These details may also help the neophytes to grasp the more involved formalism of [Bis05]. Remember that the $\hat{e}^j = \partial_{p_j}$ and $e_j = \partial_{q^j} + \Gamma_{ij}^{\ell}p_\ell\partial_{p_i}$ form a basis of T_xX while the $e^j = dq^j$ and $\hat{e}_j = dp_j - \Gamma_{ij}^{\ell}p_\ell dq^i$ form a basis of $\bigwedge^1 T_x^*X$.

When we work on Q, $(e_j = \partial_{q^j})_{1\leq j\leq}$ is a basis of T_qQ, $(e^j = dq^j)_{1\leq j\leq d}$ a basis of T_q^*Q and $(\hat{e}_j = dp_j)_{1\leq j\leq d}$ another copy of $(\partial_{q^j})_{1\leq j\leq d}$. The fiber bundle $F = (\bigwedge T^*Q)\otimes_Q(\bigwedge TQ)\otimes_Q(Q\times\mathbb{C})$ is endowed with the hermitian metric $g^F = (\bigwedge g^{-1})\otimes(\bigwedge g)$ and the Levi-Civita connection characterized by

$$\nabla^F_{\partial q^i}(dq^j) = -\Gamma^j_{i\ell}dq^\ell \quad \text{and} \quad \nabla^F_{\partial_{q^i}}(dp_j) = \Gamma^\ell_{ij}dp_\ell\,.$$

After identifying $(T_xX)^H$ with T_qQ and $(T_xX)^V$ with T_q^*Q for $x=(q,p)$, the complexified Grassman bundle $\bigwedge T^*X\otimes_X(X\times\mathbb{C})$ is nothing but $F_X = \pi^*F$ introduced in Subsection 8.4 with the natural projection $\pi : X = T^*Q\to Q$. A basis of $\bigwedge T_x^*X\otimes\mathbb{C}$ (or F) is given by $e^I\hat{e}_J = e^{i_1}\wedge\ldots\wedge e^{i_p}\wedge\hat{e}_{j_1}\wedge\ldots\wedge\hat{e}_{j_{p'}}$ with $I=\{i_1,\ldots,i_p\}$ and $J=\{j_1,\ldots,j_{p'}\}$, the i_k's and j_k's being written in the increasing order. The hermitian metric $g^{F_X} = \pi^*g^F$ is extended to the exterior algebra from

$$g^{F_X}(x;e^i,e^j) = g^{ij}(q) \quad , \quad g^{F_X}(x;\hat{e}_i,\hat{e}_j) = g_{ij}(q) \quad , \quad g^{F_X}(x;e^i,\hat{e}_j) = 0\,,$$

where $e^j = dq^j$ and $\hat{e}_j = dp_j - \Gamma^\ell_{ij}p_\ell dq^i$ at $x=(q,p)$.
According to (94) the connection ∇^{F^X} is characterized by

$$\begin{aligned} &\nabla^{F_X}_{e_i}\hat{e}_j = \Gamma^\ell_{ij}\hat{e}_\ell \quad , \quad \nabla^{F_X}_{e_i}e^j = -\Gamma^j_{i\ell}e^\ell\,,\\ \text{and}\qquad &\nabla^{F_X}_{\hat{e}^i}\hat{e}_j = 0 \quad , \quad \nabla_{\hat{e}^i}e^j = 0\,, \end{aligned}$$

and it is compatible with the metric g^{F_X} because ∇^F is compatible with g^F.

We take $b=\mp1$ and we note that the hypoelliptic Laplacian

$$\begin{aligned} 2\mathcal{U}^2_{\mp1,\mathcal{H}} = \frac{1}{2}\big[&-\Delta_p + |p|^2 + 2(\hat{e}_i\wedge)\mathbf{i}_{\hat{e}^i} - \dim Q\\ &-\frac{1}{2}\langle R^{TX}(e_i,e_j)e_k\,,\,e_\ell\rangle(e^i\wedge)(e^j\wedge)\mathbf{i}_{\hat{e}^k}\mathbf{i}_{\hat{e}^\ell}\big] \pm \mathcal{L}_{\mathcal{Y}^\mathcal{H}}\,, \end{aligned} \tag{110}$$

and we follow the presentation of [Leb1] in order to write it in the form of Definition 8.8. The Lie derivative $\mathcal{L}_{\mathcal{Y}^\mathcal{H}}$ applied to the form $\omega = \omega^J_Ie^I\wedge\hat{e}_J$ gives

$$\mathcal{L}_{\mathcal{Y}^\mathcal{H}}\omega = (\mathcal{Y}^\mathcal{H}\omega^J_I)e^I\wedge\hat{e}_J + \omega^J_I\mathcal{L}_{\mathcal{Y}^\mathcal{H}}(e^I\wedge\hat{e}_J)\,. \tag{111}$$

We compute

$$\begin{aligned} \mathcal{L}_{\mathcal{Y}^\mathcal{H}}(dq^j) &= d(\mathcal{Y}^\mathcal{H}q^j) = d(\partial_{p_j}\mathcal{H}) = (\partial^2_{q^kp_j}\mathcal{H})dq^k + (\partial^2_{p_kp_j}\mathcal{H})dp_k\,,\\ \mathcal{L}_{\mathcal{Y}^\mathcal{H}}(dp_j) &= d(\mathcal{Y}^\mathcal{H}p_j) = -d(\partial_{q^j}\mathcal{H}) = -(\partial^2_{q^kq^j}\mathcal{H})dq^k - (\partial^2_{p_kq^j}\mathcal{H})dp_k\,. \end{aligned}$$

For $\mathcal{H}=\mathcal{E}$, let us compute

$$\mathcal{L}_{\mathcal{Y}^{\mathcal{E}}}(e^j)-\nabla^{F_X}_{\mathcal{Y}^{\mathcal{E}}}e^j \quad \text{and} \quad \mathcal{L}_{\mathcal{Y}^{\mathcal{E}}}(\hat{e}_j)-\nabla^{F_X}_{\mathcal{Y}^{\mathcal{E}}}\hat{e}_j\,.$$

From $e^j=dq^j$, $\hat{e}_j=dp_j-\Gamma^i_{kj}p_idq^k$, $\Gamma^k_{ij}=\Gamma^k_{ji}$ (the Levi-Civita is torsion free) and

$$\partial_{q^\ell}g^{ij}=-g^{jk}\Gamma^i_{\ell k}-g^{i\ell}\Gamma^j_{\ell k}\,,$$

we deduce

$$\begin{aligned}
\mathcal{L}_{\mathcal{Y}^{\mathcal{E}}}e^j-\nabla^{F_X}_{\mathcal{Y}^{\mathcal{E}}}e^j &= (\partial_{q^k}g^{ij})p_idq^k+g^{jk}dp_k-g^{ik}p_i\nabla^{F_X}_{e_k}e^j\\
&= (\partial_{q^k}g^{ij})p_ie^k+g^{jk}dp_k+g^{ik}p_i\Gamma^j_{k\ell}e^\ell\\
&= g^{jk}\hat{e}_k\,.
\end{aligned} \tag{112}$$

The computation of $\mathcal{L}_{\mathcal{Y}^{\mathcal{E}}}\hat{e}_j-\nabla^{F_X}_{\mathcal{Y}^{\mathcal{E}}}\hat{e}_j$ with $\hat{e}_j=dp_j-\Gamma^i_{jk}p_idq^k$ relying on the same argument is a bit more involved:

$$\begin{aligned}
\mathcal{L}_{\mathcal{Y}^{\mathcal{E}}}\hat{e}_j-\nabla^{F_X}_{\mathcal{Y}_{\mathcal{E}}}\hat{e}_j =&-(\partial^2_{q^jq^k}g^{i\ell})p_ip_\ell dq^k-(\partial_{q^j}g^{ik})p_idp_k\\
&-\Gamma^n_{jm}p_n(\partial_{q^k}g^{im})p_idq^k-\Gamma^n_{jm}p_ng^{mk}dp_k-g^{i\ell}p_i\nabla^{F_X}_{e_\ell}\hat{e}_j\\
=&\left((\partial^2_{q^jq^k}g^{i\ell})+\Gamma^\ell_{jm}(\partial_{q^k}g^{im})\right)p_ip_\ell dq^k\\
&+g^{im}\Gamma^k_{mj}p_idp_k-g^{i\ell}p_i\Gamma^k_{\ell j}\hat{e}_k\\
=&\left((\partial^2_{q^jq^k}g^{i\ell})+\Gamma^\ell_{jm}(\partial_{q^k}g^{im})\right)p_ip_\ell dq^k\\
&+g^{i\ell}p_i\Gamma^k_{\ell j}\Gamma^n_{kn'}p_ndq^{n'}\\
=&\left((\partial^2_{q^jq^k}g^{i\ell})+\Gamma^\ell_{jm}(\partial_{q^k}g^{im})+g^{in}\Gamma^m_{nj}\Gamma^\ell_{mk}\right)p_ip_\ell e^k\,,
\end{aligned} \tag{113}$$

where the last factor of e^k can be related with the Riemann curvature tensor. The computation of $\mathcal{L}_{\mathcal{Y}^V}-\nabla^{F_X}_{\mathcal{Y}^V}$ is even simpler because $\mathcal{Y}^{\mathcal{V}}=-\partial_{q^j}V(q)\partial_{p_j}$ and $\nabla^{F_X}_{\mathcal{Y}^V}=0$ while

$$\mathcal{L}_{\mathcal{Y}^V}(dq^j)=0 \quad \text{and} \quad \mathcal{L}_{\mathcal{Y}^V}(dp_j)=-(\partial^2_{q^jq^k}V)dq^k$$

imply

$$(\mathcal{L}_{\mathcal{Y}^V}-\nabla^{F_X}_{\mathcal{Y}^V})(e^j)=0 \tag{114}$$

$$\text{and} \quad (\mathcal{L}_{\mathcal{Y}^V}-\nabla^{F_X}_{\mathcal{Y}^V})(\hat{e}_j)=\left(-(\partial^2_{q^jq^k}V)+\Gamma^\ell_{jk}(\partial_{q^\ell}V)\right)e^k\,. \tag{115}$$

The relations (111)(112)(113)(114)(111), with $\mathcal{Y}^{\mathcal{H}}=g^{ij}p_i\partial_{q^j}-(\partial_{q^j}V)\partial_{p_j}$ for $\mathcal{H}(q,p)=\frac{|p|^2_q}{2}+V(q)$, specify the action of the derivation $\mathcal{L}_{\mathcal{Y}^{\mathcal{H}}}-\nabla_{\mathcal{Y}^{\mathcal{H}}}$

on $\mathcal{C}^\infty(X;\bigwedge T^*X\otimes\mathbb{C})=\mathcal{C}^\infty(X;\pi^*F)$. If one keeps the metric g^{F_X} for the definition of L^2-sections, it is not a geometric Kramers-Fokker-Planck operator because the right-hand side (113) is homogeneous of degre 2 w.r.t p and cannot be included in the remainder term $M(q,p,\partial_p)$ of Definition 8.8. This is corrected by considering the $|p|_q$-homogeneity of $e^j=dq^j$ with degree 0 and of $\hat{e}_j=dp_j-\Gamma^i_{jk}p_idq^k$ with degree 1. Consider the weighted metric $\langle p\rangle^{2\widehat{\deg}}g^{F_X}$ given by

$$[\langle p\rangle^{2\widehat{\deg}}g^{F_X}](x;e^I\hat{e}_J,e^{I'}\hat{e}_{J'})=\langle p\rangle_q^{2|J|}g^{F_X}(x;e^I\hat{e}_J,e^{I'}\hat{e}_{J'})\,,$$

which is non zero iff $|I|=|I'|$ and $|J|=|J'|$. The local operator $\langle p\rangle^{\pm\widehat{\deg}}$ is defined by $\langle p\rangle^{\pm\widehat{\deg}}e^I\hat{e}_J=\langle p\rangle_q^{\pm|J|}e^I\hat{e}_J$ and $\langle p\rangle_q^{-\widehat{\deg}}$ is a unitary operator from $L^2(X,dqdp;(F_X,g^{F_X}))$ onto $L^2(X,dqdp;(F_X,\langle p\rangle^{2\widehat{\deg}}g^{F_X}))$. Thus studying the hypoelliptic Laplacian $2\mathcal{U}^2_{\mp1,\mathcal{H}}$ in $L^2(X,dqdp;(F_X,\langle p\rangle^{2\widehat{\deg}}g^{F_X}))$ is the same as studying

$$\mathcal{K}_{\pm,\mathcal{H}}=\langle p\rangle^{+\widehat{\deg}}\circ 2\mathcal{U}^2_{\mp1,\mathcal{H}}\circ\langle p\rangle^{-\widehat{\deg}}\tag{116}$$

in $L^2(X,dqdp;(F_X,g^{F_X}))$. After the conjugation with $\langle p\rangle^{\widehat{\deg}}$ all the right-hand sides of (112)(113)(114)(115) take the form of the remainder term $M(q,p,\partial_p)$ (remember that $\mathcal{L}_{\mathcal{Y}^H}-\nabla^{F_X}_{\mathcal{Y}^{\mathcal{H}}}$ is a derivation).
Hence $\mathcal{K}_{\pm,\mathcal{H}}$ is a geometric Kramers-Fokker-Planck operator according to Definition 8.8. General boundary problems for such operators have been studied in Subsection 8.4.

9.2.2 Review of natural boundary Witten Laplacians

Consider now a compact manifold with boundary $\overline{Q}=Q\sqcup\partial Q$. The so-called Dirichlet and Neumann boundary conditions for the Witten Laplacian acting on p-forms correspond after de Rham duality to the determination of relative homology and absolute homology groups respectively (see [ChLi][HeNi1][Lep3] [Lau]).
After introducing coordinates for which the metric g has the form $g=dq^1+m_{ij}(q^1,q')dq'^idq'^j$ according to (2) along the boundary, a general p-form can be written

$$\omega=\sum_{|I|=p\,,\,1\notin I}\omega_I(q)dq^I+\sum_{|I'|=p-1\,,\,1\notin I'}\omega_{\{1\}\cup I'}(q)dq^1\wedge dq^{I'}\,.$$

The tangential component and the normal component along ∂Q are respectively defined by

$$\begin{aligned}
\mathbf{t}\omega &= \sum_{|I|=p\,,\,1\not\in I} \omega_I(0,q')dq^I\,,\\
\mathbf{n}\omega &= \sum_{|I'|=p-1\,,\,1\not\in I'} \omega_{\{1\}\cup I'}(0,q')dq^1\wedge dq^{I'}.
\end{aligned}$$

The domain of Neumann Witten Laplacian is

$$D(\Delta_V^N)=\left\{u\in H^2(Q;\bigwedge T^*Q\otimes\mathbb{C})\,,\quad \left[\begin{array}{l}\mathbf{n}\omega=\omega\\ \mathbf{n}d_V\omega=0\end{array}\right.\right\}\,,$$

and the domain of the Dirichlet Laplacian is

$$D(\Delta_V^D)=\left\{u\in H^2(Q;\bigwedge T^*Q\otimes\mathbb{C})\,,\quad \left[\begin{array}{l}\mathbf{t}\omega=\omega\\ \mathbf{t}d_V^*\omega=0\end{array}\right.\right\}\,.$$

Note that the potential appears only in the boundary condition which involves the second trace, the trace of the (co-)differential, i.e.

$$\begin{aligned}
&\mathbf{n}d\omega+dV\wedge\omega\big|_{\partial Q}=0\quad\text{for }\Delta_V^N\,,\\
\text{and}\qquad &\mathbf{t}d^*\omega+\mathbf{i}_{\nabla f}\omega\big|_{\partial Q}=0\quad\text{for }\Delta_V^D\,.
\end{aligned}$$

Since the boundary conditions for the Kramers-Fokker-Planck equation involve only the first trace, natural boundary conditions should not depend on the potential (see a.e. the case of the specular reflection in Paragraph 9.1.5).

9.2.3 Neumann and Dirichlet boundary conditions for the hypoelliptic Laplacian on flat cylinders

We now work in the phase-space $\overline{X}=X\sqcup\partial X=X\sqcup T^*_{\partial Q}Q$ where $\overline{Q}$ is the half-space $\overline{\mathbb{R}^d}_-$ or possibly a half-cylinder $(-\infty,0]\times Q'$. The metric is assumed to be $g=1\oplus m$ with $\partial_{q^1}m=0$. Within this framework, boundary conditions for the hypoelliptic Laplacian, which should hopefully be related with Neumann and Dirichlet realization of the Witten Laplacian, are introduced with the help of a simple reflection principle. Remember that the functional analysis of the hypoelliptic Laplacian $\mathcal{U}^2_{\mp1,\mathcal{H}}$ given in (108) and (109), is made in standard L^2-spaces by using the conjugated form $\mathcal{K}_{\pm,\mathcal{H}}$ defined in (116).

According to the previous discussion we consider boundary conditions which do not depend on the potential $V(q)$. Since its contribution in the conjugated hypoelliptic Laplacian, $\mathcal{K}_{\pm,\mathcal{H}}$, is reduced to a relatively bounded perturbation of the case $V=0$, we can forget it at first.
The most natural introduction of boundary conditions for the hypoelliptic Laplacian, corresponding to Neumann and Dirichlet realization of the Hodge Laplacian ($V\equiv 0$), relies on the extension by symmetry or anti-symmetry like in Paragraph 9.1.8. In the position variable the extension by symmetry (resp. anti-symmetry), $\sigma_k(\omega)(q^1,q')$ defined for $q^1\in\mathbb{R}$, of a form $\omega(q^1,q')=\omega_I(q^1,q')dq^I$, initially defined for $q^1<0$, equals

$$\sigma_k(\omega_I dq^I)=\begin{cases}\omega(q^1,q') & \text{if } q^1<0\,,\\ (-1)^k\left[(-1)^{|\{1\}\cap I|}\omega_I(-q^1,q')dq^I\right] & \text{if } q^1>0\,,\end{cases}$$

with $k=0$ (resp. $k=1$). On the whole space the mapping $\tilde{\sigma}_k : L^2(Q'\times\mathbb{R};\bigwedge T^*(Q'\times\mathbb{R})\otimes\mathbb{C})$ is defined by

$$\tilde{\sigma}_k\left[\omega_I dq^I\right](q^1,q')=(-1)^k\left[(-1)^{|\{1\}\cap I|}\omega_I(-q^1,q')dq^I\right]\,.$$

A regular $\omega\in\mathcal{C}_0^\infty(\overline{Q},\bigwedge T^*Q\otimes\mathbb{C})$ satisfies the Neumann (resp. Dirichlet) boundary conditions, $\mathbf{n}\omega=0$ and $\mathbf{n}d\omega=0$ (resp. $\mathbf{t}\omega=0$ and $\mathbf{t}d^*\omega=0$) iff the extended form $\tilde{\eta}=\sigma_k(\eta)$ for $\eta\in\{\omega,d\omega\}$, (resp. $\eta\in\{\omega,d^*\omega\}$) has a trace along $\partial Q=\{0\}\times Q'$ (no discontinuity):

$$\sigma_k(\eta)(0^+,q')=\eta(0^-,q')\,.$$

This condition implies that the extended form $\tilde{\omega}=\sigma_k\omega$ satifies in addition to $\tilde{\sigma}_k\tilde{\omega}=\tilde{\omega}$, the continuity of $\tilde{\omega}$ and $d\tilde{\omega}$ (resp. $d^*\tilde{\omega}$) along ∂Q.
While keeping $\overline{Q}=Q'\times(-\infty,0]$ with $\partial_{q^1}m\equiv 0$ and $V\equiv 0$, the phase-space reflection extensively used in Subsection 5.3, Subsection 7.2 and Paragraph 9.1.8 is given by

$$(q^1,q',p_1,p')\to(-q^1,q',-p_1,p')\,.$$

The differentials dq^1 and dp_1 are pushed forward to $-dq^1$ and $-dp_1$ by this mapping. Hence the extension Σ_k, with the general Definition 5.3, becomes here

$$\Sigma_k(\omega)(q^1,q',p_1,p')=\omega(q^1,q',p_1,p')\qquad \text{if } q^1<0\,,$$

and

$$\Sigma_k(\omega)(q^1,q',p_1,p')=(-1)^k\left[(-1)^{|\{1\}\cap I|+|\{1\}\cap J|}\hat{\omega}_I^J(-q^1,q',-p_1,p')dq^I\wedge dp_J\right]$$

if $q^1 > 0$, when
$$\omega = \hat{\omega}_I^J dq^I \wedge dp_J\,.$$
The Neumann case corresponds to $k = 0$ and the Dirichlet case to $k = 1$. Accordingly the mapping $\tilde{\Sigma}_k$ is given by

$$\tilde{\Sigma}_k\left[\hat{\omega}_I^J dq^I \wedge dp_J\right] = (-1)^k\left[(-1)^{|\{1\}\cap I|+|\{1\}\cap J|}\hat{\omega}_I^J(-q^1, q', -p_1, p')dq^I \wedge dp_J\right]\,.$$

The corresponding boundary conditions for the conjugated hypoelliptic Laplacian $\mathcal{K}_{\pm,\mathcal{H}}$, can be written

$$\hat{\omega}_I^J(0, q', p_1, p')\big|_{p_1<0} = \begin{cases} \hat{\omega}_I^J(0, q', -p_1, p') & \text{if } |\,\{1\}\cap I| + |\,\{1\}\cap J| \text{ is even}\,,\\ -\hat{\omega}_I^J(0, q', -p_1, p') & \text{if } |\,\{1\}\cap I| + |\,\{1\}\cap J| \text{ is odd}\,,\end{cases}$$

for the Neumann boundary conditions ($k = 0$), and by

$$\hat{\omega}_I^J(0, q', p_1, p')\big|_{p_1<0} = \begin{cases} \hat{\omega}_I^J(0, q', -p_1, p') & \text{if } |\,\{1\}\cap I| + |\,\{1\}\cap J| \text{ is odd}\,,\\ -\hat{\omega}_I^J(0, q', -p_1, p') & \text{if } \{1\}\cap I| + |\,\{1\}\cap J| \text{ is even}\,,\end{cases}$$

for the Dirichlet boundary conditions ($k = 1$). Those boundary conditions extend to the case of p-forms the specular boundary condition for the scalar case reviewed in Paragraph 9.1.5 and the boundary condition with a change of sign presented in Paragraph 9.1.8.
Nevertheless they are not yet written in the proper form considered in Subsection 8.4 for $(dq^I \wedge dp_J)_{|I|\leq d, |J|\leq d}$ is not the basis $(e^I\hat{e}_J)_{|I|\leq d,|J|\leq d}$ corresponding to the identification $\bigwedge T^*X = F_X$ with $F = (\bigwedge T^*Q) \otimes_Q (\bigwedge TQ)$. Fortunately, our assumption $g = 1 \oplus m$ with $\partial_{q^1} m = 0$ imply

$$\Gamma^i_{1j} = \Gamma^i_{j1} = \Gamma^1_{ij} = 0\,,$$

which means $e^1 = dq^1$, $\hat{e}_1 = dp_1$ and $\mathbf{i}_{\partial_{q^1}} e^I\hat{e}_J = \mathbf{i}_{\partial_{p_1}} e^I\hat{e}_J = 0$ when $1 \notin I$ and $1 \notin J$. Therefore the forms $\Sigma_k(\omega)1_{\mathbb{R}_+}(q_1) = \Sigma_k(\omega_I^J e^I\hat{e}_J)1_{\mathbb{R}_+}(q_1)$ and $\tilde{\Sigma}_k(\omega) = \tilde{\Sigma}_k(\omega_I^J e^I\hat{e}_J)$ are respectively equal to

$$\begin{aligned} &(-1)^k\left[(-1)^{|\{1\}\cap I|+|\{1\}\cap J|}\omega_I^J(-q^1, q', -p_1, p')e^I\hat{e}_J\right]1_{\mathbb{R}_+}(q^1)\\ \text{and}\qquad &(-1)^k\left[(-1)^{|\{1\}\cap I|+|\{1\}\cap J|}\omega_I^J(-q^1, q', -p_1, p')e^I\hat{e}_J\right]\,.\end{aligned}$$

Hence the Neumann ($k = 0$) boundary conditions now written as

$$\omega_I^J(0, q', p_1, p')\big|_{p_1<0} = \begin{cases} \omega_I^J(0, q', -p_1, p') & \text{if } |\,\{1\}\cap I| + |\,\{1\}\cap J| \text{ is even}\,,\\ -\omega_I^J(0, q', -p_1, p') & \text{if } \{1\}\cap I| + |\,\{1\}\cap J| \text{ is odd}\,,\end{cases} \tag{117}$$

and the Dirichlet ($k=1$) boundary conditions now written as

$$\omega_I^J(0,q',p_1,p')\big|_{p_1<0} = \begin{cases} \omega_I^J(0,q',-p_1,p') & \text{if } |\{1\}\cap I| + |\{1\}\cap J| \text{ is odd}\,, \\ -\omega_I^J(0,q',-p_1,p') & \text{if } \{1\}\cap I| + |\{1\}\cap J| \text{ is even}\,, \end{cases} \tag{118}$$

correspond to the case

$$\begin{aligned} & \mathbf{A} = 0\,, \\ \text{and} \quad & \mathbf{j}(e^I\hat{e}_J) = (-1)^k(-1)^{|\{1\}\cap I|+|\{1\}\cap J|}e^I\hat{e}_J \end{aligned}$$

of Proposition 8.10 with respectively $k=0$ for and $k=1$. We shall denote $\mathcal{K}_{\pm,\mathcal{H}}^N$ (resp. $\mathcal{K}_{\pm,\mathcal{H}}^D$) the Neumann (resp. Dirichlet) realization corresponding to $k=0$ (resp. $k=1$) of $\mathcal{K}_{\pm,\mathcal{H}}$. Proposition 8.10 applies to $\mathcal{K}_{\pm,\mathcal{H}}^N$ and $\mathcal{K}_{\pm,\mathcal{H}}^D$ also when $V(q)\neq 0$ is a globally Lipschitz potential.

We end this section by checking the commutation with the differential on a dense set of the domain. The hypoelliptic Laplacian $\mathcal{U}_{\mp,\mathcal{H}}$ as a differential operator acting on $\mathcal{C}^\infty(\overline{X};F_X)$ commutes with $d_{\mathcal{H}}^X$. Equivalently $\mathcal{K}_{\pm,\mathcal{H}}$ commutes with

$$\langle p\rangle^{\widehat{\deg}} d_{\mathcal{H}}^X \langle p\rangle^{-\widehat{\deg}} = \langle p\rangle^{\widehat{\deg}}(e^{-\mathcal{H}}d^X e^{\mathcal{H}})\langle p\rangle^{-\widehat{\deg}} = \langle p\rangle^{\widehat{\deg}}(d^X + d\mathcal{H}\wedge)\langle p\rangle^{-\widehat{\deg}}\,.$$

Proposition 9.3. *Let $\mathcal{K}_{\pm,\mathcal{H}}^N$ (resp. $\mathcal{K}_{\pm,\mathcal{H}}^D$) be the Neumann (resp. Dirichlet) realization of $\mathcal{K}_{\pm,\mathcal{H}}$ when $\overline{Q} = (-\infty,0]\times Q'$ and $\partial_{q^1}m\equiv 0$ (Q' is either compact or a compact perturbation of the euclidean space $\mathbb{R}^{d-1}$) and $\mathcal{H}(q,p) = \frac{|p|_q^2}{2} + V(q)$ and $\mathcal{V}$ globally Lipschitz on $\overline{Q}$. Then the set $\mathcal{D}$ of sections $\omega = \omega_I^J e^I\hat{e}_J \in \mathcal{C}_0^\infty(\overline{X};F_X)$ which satisfy (117) (resp. (118)) and*

$$|\partial_{q^1}[e^{\mathcal{H}(q,p)}\langle p\rangle_q^{-\widehat{deg}}\omega](q,p)| = \mathcal{O}(|q^1|^\infty)$$

is dense in $D(K_{\pm,\mathcal{H}}^N)$ (resp. $D(K_{\pm,\mathcal{H}}^D)$) endowed with its graph norm.
For any $\omega\in\mathcal{D}$, $\langle p\rangle^{\widehat{deg}} d_{\mathcal{H}}^X\langle p\rangle^{-\widehat{deg}}\omega$ belongs to $D(K_{\pm,\mathcal{H}}^N)$ (resp. $D(K_{\pm,\mathcal{H}}^D)$) and

$$K_{\pm,\mathcal{H}}^N\langle p\rangle^{\widehat{deg}} d_{\mathcal{H}}^X\langle p\rangle^{-\widehat{deg}}\omega = \langle p\rangle^{\widehat{deg}} d_{\mathcal{H}}^X\langle p\rangle^{-\widehat{deg}}\mathcal{K}_{\pm,\mathcal{H}}^N\omega\,, \tag{119}$$

$$\textit{resp.} \qquad K_{\pm,\mathcal{H}}^D\langle p\rangle^{\widehat{deg}} d_{\mathcal{H}}^X\langle p\rangle^{-\widehat{deg}}\omega = \langle p\rangle^{\widehat{deg}} d_{\mathcal{H}}^X\langle p\rangle^{-\widehat{deg}}\mathcal{K}_{\pm,\mathcal{H}}^D\omega\,. \tag{120}$$

Proof. First of all, multiplying ω by $e^{\pm\mathcal{H}(q,p)} = e^{\pm\frac{|p|_q^2\pm V(q)}{2}}$ or $\langle p\rangle^{\pm\widehat{\deg}}$ does not affect the boundary conditions (117) and (118) because $|(-p_1,p')|_q =$

$|(p_1, p')|_q$.
The analysis of both cases, Dirichlet and Neumann, follow the same lines and we focus on the Neumann case.
Let us consider the density of $\mathcal{D}$ in $D(\mathcal{K}^N_{\pm,\mathcal{H}})$. We already know that the set of $\omega \in \mathcal{C}^\infty_0(\overline{X}; F_X)$ which satisfy the boundary conditions (117) is dense in $D(K^N_{\pm,\mathcal{H}})$. Hence it suffices to approximate compactly supported elements of $D(K^N_{\pm,\mathcal{H}})$ by elements of $\mathcal{D}$. When $\chi \in \mathcal{C}^\infty_0(\overline{X}; \mathbb{R})$ is a cut-off function such that $\chi(0, q', -p_1, p') = \chi(0, q', p_1, p')$, one has

$$e^{+\mathcal{H}} \langle p \rangle^{-\widehat{deg}} \mathcal{K}^N_{\pm,\mathcal{H}} e^{-\mathcal{H}} \langle p \rangle^{\widehat{deg}} \chi = \left[\mathcal{K}^N_{\pm,\mathcal{H}} + M_\chi(q, p, \partial_p)\right] \chi \, ,$$

where $M_\chi(q, p, \partial_p)$ satisfies the conditions (95) and (96) of Definition 8.8. Like in the proof of Proposition 8.10, $\mathcal{K}^N_{\pm,\mathcal{H}}$ can be identified locally and up to an additional correcting term $M(q, p, \partial_p)$, with $\tilde{U}(q) K^{g^{\mathfrak{f}}}_{\pm,0,g} \tilde{U}(q)$ where $\tilde{U}$ is a unitary transform $\tilde{U}(q)$ from $L^2(X, dqdp; (\mathfrak{f}, g^{\mathfrak{f}}))$ to $L^2(X, dqdp; F_X)$. We can additionally assume $\partial_{q^1} \tilde{U}(q) \equiv 0$ in our case where $\partial_{q^1} m \equiv 0$. With those local transformations (introduce also finite partition in the position variable of unity in order to reduce the problem to local ones), the density actually amounts to the density of $\mathcal{D}(\overline{X}, j)$ in $D(K^{g^{\mathfrak{f}}}_{\pm,0,g})$ stated in Theorem 1.1 .
Assume now that ω belongs to $\mathcal{D}$. Then $\eta = e^{\mathcal{H}(q,p)} \langle p \rangle^{-\widehat{\deg}} \omega$ belongs to $\mathcal{C}^\infty_0(\overline{X}; F_X)$, satisfies the boundary conditions (117) and

$$|\partial_{q^1} \eta(q, p)| = \mathcal{O}(|q^1|^\infty) \, .$$

The extension by symmetry $\tilde{\eta} = \Sigma_1(\eta)$ satisfies $\tilde{\Sigma}_1(\tilde{\eta}) = \tilde{\eta}$. We deduce $\tilde{\Sigma}_1(d^X \tilde{\eta}) = d^X \tilde{\Sigma}(\tilde{\eta}) = d^X \tilde{\eta}$ outside $\{q^1 = 0\}$ while the above vanishing condition ensures that $d^X \tilde{\eta}$ has no discontinuity at $\{q^1 = 0\}$. We deduce that $d^X \eta$ and therefore $\langle p \rangle^{\widehat{\deg}} d^X_{\mathcal{H}} \langle p \rangle^{-\widehat{\deg}} \omega = \langle p \rangle^{\widehat{\deg}} d^X \eta$ fulfill the boundary conditions (117). We have proved that $\langle p \rangle^{\widehat{\deg}} d^X_{\mathcal{H}} \langle p \rangle^{-\widehat{\deg}} \omega$ belongs to $D(K^N_{\pm,\mathcal{H}})$. The commutation (119) is now nothing but the known commutation of differential operators. $\square$

9.2.4 Neumann and Dirichlet realizations of the hypoelliptic Laplacian

We consider now the general case when $\overline{Q}$ is either a compact manifold with boundary or a compact perturbation of the euclidean half-space $\overline{\mathbb{R}^d_-}$. The hamiltonian function $\mathcal{H}$ is given by

$$\mathcal{H}(q,p) = \mathcal{E}(q,p) + V(q) = \frac{|p|_q^2}{2} + V(q)$$

where $V(q)$ is a globally Lipschitz function.
Remember that $\mathcal{K}_{\pm,\mathcal{H}}$ is a geometric Kramers-Fokker-Planck operator according to Definition 8.8. After introducing the basis $e^I\hat{e}_J$ of $F_X = \bigwedge T^*X$ like in Paragraph 9.2.1 with $e^i = dq^i$ and $\hat{e}_j = dp_j - \Gamma_{ij}^{\ell}p_\ell dq^i$, our proposal of Neumann (resp. Dirichlet) boundary condtions for $\mathcal{K}_{\pm,\mathcal{H}}$ is the same as in the flat case:

$$\begin{aligned} &\omega_I^J(0,q',p_1,p')\big|_{p_1<0} = (-1)^{k+|\{1\}\cap I|+|\{1\}\cap J|}\omega_I^J(0,q',-p_1,p') \quad (121)\\ \text{when}\quad &\omega = \omega_I^J e^I\hat{e}_J\,,\end{aligned}$$

with $k=0$ (resp. $k=1$).
It still corresponds to the case

$$\begin{aligned} &\mathbf{A} = 0\,,\\ \text{and}\quad &\mathbf{j}(e^I\hat{e}_J) = (-1)^k(-1)^{|\{1\}\cap I|+|\{1\}\cap J|}e^I\hat{e}_J\end{aligned}$$

of Definition 8.9 and all the results of Proposition 8.10 with $\mathbf{A}=0$ apply to the corresponding realization $\mathcal{K}_{\pm,\mathcal{H}}^N$ (resp. $\mathcal{K}_{\pm,\mathcal{H}}^D$) of $\mathcal{K}_{\pm,\mathcal{H}}$. This can be translated directly in terms of the hypoelliptic Laplacian $\mathcal{U}_{\mp,\mathcal{H}}^2$ because applying the weight $\langle p\rangle^{\pm\widehat{\deg}}$ does not affect the boundary condition (121).

Proposition 9.4. *Let $[\mathcal{U}_{\mp1,\mathcal{H}}^2]^N$ (resp. $[\mathcal{U}_{\mp,\mathcal{H}}^2]^D$) be the closure in the Hilbert space $L^2(X,dqdp;(F_X,\langle p\rangle^{2\widehat{deg}}g^{F_X}))$ of $\mathcal{U}_{\mp1,\mathcal{H}}^2 = (\overline{d}_{\mp1,\mathcal{H}}^X + d_{\mathcal{H}}^X)^2$ initially defined with the domain*

$$\mathcal{D}_0 = \left\{\omega = \omega_I^J e^I\hat{e}_J \in \mathcal{C}_0^\infty(\overline{X};F^X)\,, (121)\ holds\right\}$$

with $k=0$ (resp. $k=1$). Then there exists $C\in\mathbb{R}$ such that $C+[\mathcal{U}_{\mp1,\mathcal{H}}^2]^N$ (resp. $C+[\mathcal{U}^2]^D$) is maximal accretive. Moreover all the results and estimates of Proposition 8.10 with $\mathbf{A}=0$ apply to $\mathcal{K}_{\pm,\mathcal{H}}^{N,D} = \langle p\rangle^{\widehat{deg}}2[\mathcal{U}_{\mp1,\mathcal{H}}^2]^{N,D}\langle p\rangle^{-\widehat{deg}}$ and are easily translated in $L^2(X,dqdp;(F_X,\langle p\rangle^{2\widehat{deg}}g^{F_X}))$ by conjugating with the unitary weight $\langle p\rangle^{\widehat{deg}}$.

Proof. The fact that $\langle p\rangle^{-\widehat{\deg}}\mathcal{K}_{\pm,\mathcal{H}}^{N,D}\langle p\rangle^{\widehat{\deg}}$ is exactly the closure of $2\mathcal{U}_{\mp 1,\mathcal{H}}^{2}$ with the domain $\mathcal{D}_0$ comes from the density of $\mathcal{C}_0^\infty(\overline{X};F_X)$ in $D(\mathcal{K}_{\pm,\mathcal{H}}^{N,D})$ endowed with its graph norm. $\square$

We end this paragraph with two questions:

1. The accurate spectral analysis of boundary Witten Laplacians developed in [ChLi][HelNi][Lep3] relies strongly on the commutations

$$\begin{array}{ll} (z-\Delta_V^{D,N})^{-1}d_V\omega = d_V(z-\Delta_V^{D,N})^{-1} & , \\ (z-\Delta_V^{D,N})^{-1}d_V^*\omega = d_V^*(z-\Delta_V^{D,N})^{-1} & , \end{array}$$

   for all $z\notin\sigma(\Delta_V^{D,N})$ and all ω belonging to the form domain of $\Delta_V^{D,N}$. Is there a dense set in $L^2(X,dqdp;(F_X,\langle p\rangle^{\widehat{\deg}}g^{F_X})$ in which the commutation

$$(z-\mathcal{U}_{\mp 1,\mathcal{H}}^{2})^{-1}d_{\mathcal{H}}^{X} = d_{\mathcal{H}}^{X}(z-\mathcal{U}_{\mp 1,\mathcal{H}}^{2})^{-1}\,,$$

   holds ?

2. Are the names Neumann and Dirichlet relevant ? In particular by introducing the parameter b in $\mathcal{U}_{b,\mathcal{H}}^{2}$, does the operator $\mathcal{U}_{b,\mathcal{H}}^{N}$ (resp. $\mathcal{U}_{b,\mathcal{H}}^{D}$) converge in some sense to Δ_V^N (resp. Δ_V^D) as $b\to 0$ (large friction limit) ?

A Translation invariant model problems

In this appendix we review a few properties of the operator

$$P_\pm = \pm p.\partial_q + \frac{-\Delta_p+|p|^2}{2} = \sum_{j=1}^{d} \pm p_j\partial_{q^j} + \frac{-\partial_{p_j}^2+p_j^2}{2} \quad \text{on } T^*Q\,,$$

where $Q=\mathbb{R}^{d'}\times\mathbb{T}^{d''}$, $d',d''\in\mathbb{N}$, $\mathbb{T}^{d''}=\mathbb{R}^{d''}/\mathbb{Z}^{d''}$, and its cotangent bundle $X=T^*Q=\mathbb{R}^{d'}\times\mathbb{T}^{d''}\times\mathbb{R}^{d}$, $d=d'+d''$, are endowed with euclidean metrics. The operator $P_\pm$ is a typical example of a type II hypoelliptic operator (see [Hor67, HelNi, Kol]) in the sense that $P_\pm u\in\mathcal{C}^\infty(X)$ implies $u\in\mathcal{C}^\infty(X)$. More explicit subelliptic and pseudospectral estimates can be given. This appendix collects a few of them and we refer to [HerNi] and [HelNi] for results with a potential, to [HiPr09][Pra] for more general results concerned with differential operators with quadratic symbols and to [Leb1, Leb2] for geometric Kramers-Fokker-Planck operators (see also Section 6).

Remember the notation $\mathcal{H}^{s'}=\left\{u\in\mathcal{S}'(\mathbb{R}^d)\,,(\frac{d}{2}+\mathcal{O})^{\frac{s'}{2}}u\in L^2(\mathbb{R}^d)\right\}$ with $\mathcal{O}=\frac{-\Delta_p+|p|^2}{2}$, and remember that

$$\|\big(\frac{-\Delta_p+|p|^2+d}{2}\big)^N u\|_{L^2}\quad\text{and}\quad\sum_{|\alpha|+|\beta|=2N}\|p^\alpha\partial_p^\beta u\|_{L^2}$$

are equivalent norms on $\mathcal{H}^{2N}$ when $s'=2N\in2\mathbb{N}$, owing to the global ellipticity of $\mathcal{O}+\frac{d}{2}$ (see [Hel1][HelNi] or [HormIII]-Chap 18). The operator $\mathcal{O}$ is diagonal in the Hermite basis $(\varphi_\nu)_{\nu\in\mathbb{N}^d}$ given, with the multi-index notation, by

$$\begin{aligned}
&\varphi_\nu=\frac{1}{\sqrt{\nu!}}(a^*)^\nu\varphi_0\quad,\quad\varphi_0=\frac{e^{-\frac{|p|^2}{2}}}{\pi^{\frac{d}{4}}}\,,\\
&a=\frac{\partial_p+p}{\sqrt{2}}\quad,\quad a^*=\frac{-\partial_p+p}{\sqrt{2}}\,,\\
&(\mathcal{O}-\frac{d}{2})\varphi_\nu=a^*a\varphi_\nu=|\nu|\varphi_\nu\,.
\end{aligned}$$

One easily checks with this Hermite basis, $\cap_{s'\in\mathbb{R}}\mathcal{H}^{s'}=\mathcal{S}(\mathbb{R}^d)$ and $\cup_{s'\in\mathbb{R}}\mathcal{H}^{s'}=\mathcal{S}'(\mathbb{R}^d)$. We shall also use the vertical weighted L^2-spaces

$$L^2_{s'}=\left\{u\in\mathcal{S}'(\mathbb{R}^d)\,,\langle p\rangle^{s'}u\in L^2(\mathbb{R}^d)\right\}\,.$$

In all variables $(q,p)=(q,q',p)$, the Schwartz space $\mathcal{S}(X)$ is the set $\mathcal{C}^\infty$-functions u such that

$$\forall(\alpha,\beta)\in\mathbb{N}^{d_1+d}\times\mathbb{N}^{2d}\,,\ \sup_{(q,p)\in X}\left|(q',p)^\alpha\partial^\beta_{(q,p)}u(q,p)\right|<+\infty\,,$$

and its dual is denoted by $\mathcal{S}'(X)$. By using the Fourier transform in $q'\in\mathbb{R}^d$ and Fourier series in the variable $q''\in\mathbb{T}^{d''}$,

$$\hat{u}(\xi',\xi'',p)=\int_{\mathbb{R}^{d'}\times\mathbb{T}^{d''}}e^{-i\xi.q}u(q,p)\,dq\,,\quad\xi=(\xi',\xi'')\in\mathbb{R}^{d'}\times(2\pi\mathbb{Z})^{d''}\,,$$

the norms $(p_N)_{N\in\mathbb{N}}$ defined by

$$p_N(u)=\sqrt{\sum_{|\alpha|+|\beta|+m'\leq N}\|\xi^\alpha\partial^\beta_{\xi'}\hat{u}\|^2_{L^2(\mathbb{R}^{d'}\times(2\pi\mathbb{Z})^{d''};\mathcal{H}^{m'})}}\,,\tag{122}$$

provide to $\mathcal{S}(X)$ its Fréchet-space structure.
For $\mathcal{E}=\mathcal{H}^{s'}$ or $\mathcal{E}=L^2_{s'}$, the Sobolev spaces of $\mathcal{E}$-valued functions are defined by

$$H^s(Q;\mathcal{E})=\left\{u\in\mathcal{S}'(X)\,,(1-\Delta_q)^{\frac{s}{2}}u\in L^2(Q,dq;\mathcal{E})\right\}\,.$$

When $d'=0$ (and only in this case), we know $\mathcal{S}(X)=\cap_{s,s'\in\mathbb{R}}H^s(Q;\mathcal{H}^{s'})$, $\mathcal{S}'(X)=\cup_{s,s'\in\mathbb{R}}H^s(Q;\mathcal{H}^{s'})$ and the embedding $H^{s_1}(Q;\mathcal{H}^{s'_1})\subset H^{s_2}(Q;\mathcal{H}^{s'_2})$ is compact as soon as $s_1>s_2$ and $s'_1>s'_2$.

Theorem A.1. *The operator $K_\pm-\frac{d}{2}$ defined in $L^2(X,dqdp)$ by*

$$\begin{aligned}&D(K_\pm)=\left\{u\in L^2(X,dqdp),P_\pm u\in L^2(X,dqdp)\right\}\\&\forall u\in D(K_\pm)\,,\quad K_\pm u=P_\pm u\,,\end{aligned}$$

is maximal accretive. The spaces $\mathcal{S}(X)$ and $\mathcal{C}_0^\infty(X)$ are dense in $D(K_\pm)$ endowed with its graph norm.
For any $\lambda\in\mathbb{R}$ the operator $(P_\pm-i\lambda)$ is an automorphism of $\mathcal{S}(X)$ and of $\mathcal{S}'(X)$. For any $(\lambda,s,s')\in\mathbb{R}^3$, the quantities

$$\|\big(\frac{-\Delta_p+|p|^2+d}{2}\big)u\|_{H^s(Q;L^2_{s'})}\quad,\quad\|u\|_{H^{s+\frac{2}{3}}(Q;L^2_{s'})}\quad\text{and}\quad\langle\lambda\rangle^{\frac{1}{2}}\|u\|_{H^s(Q;L^2_{s'})}\,,\tag{123}$$

are bounded by $C_{s'}\|(P_\pm-i\lambda)u\|_{H^s(Q;L^2_{s'})}$ with $C_{s'}$ independent of (s,λ).
In particular with $s,s'=0$, this implies

$$D(K_\pm)=D(K_\mp)=\left\{u\in L^2(X,dqdp)\,,\quad p.\partial_q u\,,\,\mathcal{O}u\in L^2(X,dqdp)\right\}\,.$$

The adjoint of $K_\pm$ is $K_\pm^*=K_\mp$.

Proof. The accretivity of $P_\pm-\frac{d}{2}$ on $\mathcal{S}(X)$ comes from the direct integration by parts:

$$\mathbb{R}\mathrm{e}\,\langle u\,,\,P_\pm u\rangle=\langle u\,,\,\mathcal{O}u\rangle\geq\frac{d}{2}\|u\|^2\,,$$

for all $u\in\mathcal{S}(X)$. The maximal accretivity of $K_\pm$ comes from the fact that $(P_\pm-i\lambda)$ is an automorphism of $\mathcal{S}'(X)$. The space $\mathcal{S}(X)$ is dense in $D(K_\pm)$ because $(P_\pm-i\lambda)$ is an automorphism of $\mathcal{S}(X)$. The density of $\mathcal{C}_0^\infty(X)$ is obtained after truncating elements of $\mathcal{S}(X)$ while $P_\pm$ is a differential operator with polynomial coefficients.
Thus the problem amounts to solving $(P_\pm-i\lambda)u=f$ in $\mathcal{S}'(X)$ and to study the regularity properties of u according to f. Due to the translational

invariance in q and after a Fourier transform (series in ξ''), this is reduced to solving and finding parameter dependent estimates for the equation

$$(P_{\pm\xi} - i\lambda)\hat{u}(\xi, p) = \hat{f}(\xi, p), \quad \xi = (\xi', \xi'') \in \mathbb{R}^{d'} \times (2\pi\mathbb{Z})^{d''},$$

with

$$P_\xi = ip.\xi - \frac{\Delta_p + |p|^2}{2}.$$

Since the topology of $\mathcal{S}(X)$ is given by the countable family of norms $(p_N)_{N\in\mathbb{N}}$ given by (122) the invertibility of $P_\pm - i\lambda$ in $\mathcal{S}(X)$ is a consequence of Lemma A.2. The invertibility in $\mathcal{S}'(X)$ is deduced by duality with $P_\pm^* = P_\mp$. The estimates of the quantities (123) are ξ-integrated versions of Lemma A.3. For the adjoint the identity

$$\langle u\,,\, K_\pm v\rangle = \langle K_\mp u\,,\, v\rangle$$

holds for any $u, v \in \mathcal{S}(X)$. By the density of $\mathcal{S}(X)$ in $D(K_\pm)$ this means that $u \in \mathcal{S}(X)$ belongs to $D(K_\pm^*)$ with $K_\pm^* u = K_\mp u$. Since $\mathcal{S}(X)$ is dense in $D(K_\mp)$, this implies $K_\mp \subset K_\pm^*$. Both operators $K_\mp$ and $K_\pm^*$ are maximal accretive. The inclusion is therefore an equality. $\square$

Let us first consider the resolvent $(P_\xi - i\lambda)^{-1}$ in the $\mathcal{H}^{s'}$-space, $s' \in \mathbb{R}$.

Lemma A.2. *For $\xi \in \mathbb{R}^{d'} \times (2\pi\mathbb{Z})^{d''}$ and $\lambda \in \mathbb{R}$, the operator $P_\xi - i\lambda = i(p.\xi - \lambda) + \frac{-\Delta_p + p^2}{2}$ is an automorphism of $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$. For any $s' \in \mathbb{R}$ and $\alpha \in \mathbb{N}^{d_1}$ there exists $C_{s',\alpha} > 0$ such that*

$$\begin{aligned}(1 + |\xi|^{\frac{2}{3}} + |\lambda|^{\frac{1}{2}})\|\partial_{\xi'}^\alpha (P_\xi - i\lambda)^{-1}\|_{\mathcal{L}(\mathcal{H}^{s'})} + \|\partial_{\xi'}^\alpha (P_\xi - i\lambda)^{-1}\|_{\mathcal{L}(\mathcal{H}^{s'};\mathcal{H}^{s'+2+|\alpha|})} \\ \leq C_{\alpha,s}\langle\xi\rangle^{|s'|(1+|\alpha|)},\end{aligned}$$

holds for all $(\xi, \lambda) \in \mathbb{R}^{d'} \times (2\pi\mathbb{Z})^{d''} \times \mathbb{R}$.

Proof. **a)** Consider first the case $\alpha = 0$ and $s' = 0$: The operator $P_\xi - \frac{d}{2}$ with domain $D(P_\xi) = \mathcal{H}^2$ is maximal accretive and $(P_\xi - i\lambda)^{-1} \in \mathcal{L}(L^2(\mathbb{R}^d, dp); \mathcal{H}^2)$ for all $\xi \in \mathbb{R}^{d'} \times (2\pi\mathbb{Z})^{d''}$. In order to prove the resolvent estimates, take $u \in D(P_\xi)$ and compute

$$\|(P_\xi - i\lambda)u\|^2 = \|(p.\xi - \lambda)u\|^2 + \|(\frac{-\Delta_p + |p|^2}{2})u\|^2 + \langle u\,,\, \xi.D_p u\rangle, \quad (124)$$

with $D_p = \frac{1}{i}\partial_p$ and $\|\ \| = \|\ \|_{L^2(\mathbb{R}^d, dp)}$. With a rotation in the variable p, the problem is reduced to the case $\xi = (\xi_1, 0, \ldots, 0) \in \mathbb{R}^d$. The separation of variables $p = (p_1, p') \in \mathbb{R} \times \mathbb{R}^{d-1}$ gives

$$\begin{aligned}\frac{-\Delta_p + |p|^2}{2} &= \frac{D_{p_1}^2 + p_1^2}{2} + \frac{-\Delta_{p'} + |p'|^2}{2} \\ &= \sum_{\nu \in \mathbb{N}^{d-1}} \left[\frac{D_{p_1}^2 + p_1^2}{2} + \frac{d-1}{2} + |\nu|\right] |\varphi_\nu\rangle\langle\varphi_\nu|\,.\end{aligned}$$

where $(\varphi_\nu)_{\nu\in\mathbb{N}^{d-1}}$ is the Hermite basis in $L^2(\mathbb{R}^{d-1}, dp)$. After writing $u = \sum_{\nu\in\mathbb{N}^{d-1}} u_\nu(\xi_1, p_1)\varphi_\nu$ the problem is reduced to finding lower bounds of

$$\|(p_1\xi_1 - \lambda)u_\nu\|^2 + \|(\frac{D_{p_1}^2 + p_1^2}{2} + \frac{d-1}{2} + |\nu|)u_\nu\|^2 + \langle u_\nu\,,\, \xi_1 D_{p_1} u_\nu\rangle\,,$$

which is a one dimensional problem parametrized by (ξ_1, λ, ν). The identity

$$(D_{p_1}^2 + p_1^2 + a)^2 = D_{p_1}^4 + p_1^4 + 2a(D_{p_1}^2 + p_1^2) + a^2 + 2D_{p_1}p_1^2 D_{p_1} - 2 \quad , \quad a \geq 0\,,$$

gives

$$\begin{aligned}\|\big(\frac{D_{p_1} + p_1^2}{2} + \frac{d-1}{2} + |\nu|\big)u_\nu\|^2 \geq \|\frac{D_{p_1}^2}{2}u_\nu\|^2 + \|\frac{p_1^2}{2}u_\nu\|^2 \\ + \left[(\frac{d-1}{2} + |\nu|)^2 - \frac{1}{2}\right]\|u_\nu\|\,.\end{aligned}$$

We are thus looking for a lower bound of

$$\begin{aligned}\|(p_1\xi_1 - \lambda)u_\nu\|^2 + \|\frac{D_{p_1}^2}{2}u_\nu\|^2 + \xi_1\langle u_\nu\,,\, D_{p_1}u_\nu\rangle + \|\frac{p_1^2}{2}u_\nu\|^2 \\ + \left[(\frac{d-1}{2} + |\nu|)^2 - \frac{1}{2}\right]\|u_\nu\|^2\,. \quad (125)\end{aligned}$$

For $\xi_1 \neq 0$ (the case $\xi_1 = 0$ is obvious), set

$$u_\nu(\xi_1, p_1) = |\xi_1|^{\frac{1}{6}}\phi(\xi_1, \mathrm{sign}\,(\xi_1)|\xi_1|^{\frac{1}{3}}p_1 - |\xi_1|^{-\frac{2}{3}}\lambda)\,,$$

corresponding to the change of variable $t = \mathrm{sign}\,(\xi_1)|\xi_1|^{\frac{1}{3}}p_1 - |\xi_1|^{-\frac{2}{3}}\lambda$, and compute the first three terms

$$
\begin{aligned}
&\|(p_1\xi_1 - \lambda)u_\nu\|^2 + \|\frac{D_{p_1}^2}{2}u_\nu\|^2 + \xi_1\langle u_\nu\,,\,D_{p_1}u_\nu\rangle \\
&\qquad = |\xi_1|^{\frac{4}{3}}\left[\|t\phi\|^2 + \|\frac{D_t^2}{2}\phi\|^2 + \mathrm{sign}\,(\xi_1)\langle\phi\,,D_t\phi\rangle\right] \\
&\qquad\qquad = \frac{|\xi_1|^{\frac{4}{3}}}{4}\|(\mathrm{sign}\,(\xi_1)2it + D_t^2)\phi\|^2\,.
\end{aligned}
$$

On $L^2(\mathbb{R}_t, dt)$, the complex Airy operator $D_t^2 + \mathrm{sign}\,(\xi_1)2it$ has a compact resolvent because $4\|f\|^2 + \|(D_t^2 + \mathrm{sign}\,(\xi_1)2it)f\|^2 \geq \|2tf\|_{L^2}^2 + \|D_t^2 f\|^2$ and it is injective $(D_t^2 + \mathrm{sign}\,(\xi_1)2it)f = 0$ (or equivalently $(\tau^2 + \mathrm{sign}\,(\xi_1)2\partial_\tau)\hat{f} = 0$) has no solution in $L^2(\mathbb{R}, dt)$. We deduce

$$
\|(D_t^2 + \mathrm{sign}\,(\xi_1)2it)\phi\|^2 \geq C^{-1}\left[\|\phi\|^2 + \|2t\phi\|^2 + \|D_t^2\phi\|^2\right]\,.
$$

and

$$
\begin{aligned}
&\|(p_1\xi_1 - \lambda)u_\nu\|^2 + \|\frac{D_{p_1}^2}{2}u_\nu\|_{L^2}^2 + \xi_1\langle u_\nu\,,\,D_{p_1}u_\nu\rangle \\
&\qquad \geq \frac{1}{C'}\left[|\xi_1|^{\frac{4}{3}}\|u_\nu\|^2 + \|(p_1\xi_1 - \lambda)u_\nu\|^2 + \|D_{p_1}^2 u_\nu\|^2\right]\,, \qquad (126)
\end{aligned}
$$

with a uniform constant $C' \geq 1$. By summing over $\nu \in \mathbb{N}^{d-1}$, (124), (125) and (126) lead to

$$
\begin{aligned}
C'\|(P_\xi - i\lambda)u\|^2 &\geq \|(p.\xi - \lambda)u\|_{L^2}^2 + \|D_{p_1}^2 u\|_{L^2}^2 + \|p_1^2 u\|^2 \\
&\quad + \||\xi|^{\frac{2}{3}}u\|_{L^2}^2 + \|(\frac{-\Delta_{p'} + |p'|^2 + d - 1}{2})u\|^2 - \frac{1}{2}\|u\|^2\,.
\end{aligned}
$$

With

$$
\|u\|\|(P_\xi - i\lambda)u\| \geq \mathrm{Re}\,\langle u\,,\,(P_\xi - i\lambda)u\rangle \geq \frac{d}{2}\|u\|^2\,,
$$

which gives $\|(P_\xi - i\lambda)u\|^2 \geq \frac{d^2}{4}\|u\|_{L^2}^2$, we obtain

$$
C''\|(P_\xi - i\lambda)u\|_{L^2}^2 \geq (1 + |\xi|^{4/3})\|u\|_{L^2}^2 + \|u\|_{\mathcal{H}^2}^2 + \|(p.\xi - \lambda)u\|_{L^2}^2\,.
$$

For $|\lambda|\|u\|_{L^2}^2$, simply use

$$
\begin{aligned}
|\lambda|\|u\|_{L^2}^2 &= i\,\mathrm{sign}\,(\lambda)\langle u\,,\,i(p.\xi - \lambda)u\rangle + \mathrm{sign}\,(\lambda)\langle u\,,\,p.\xi u\rangle \\
&\leq \|u\|\|(p.\xi - \lambda)u\| + \int_{\mathbb{R}}|\xi||p||u(p)|^2\,dp
\end{aligned}
$$

with $|\xi||p| \leq C_1(|\xi|^{4/3}+|p|^4)$ in the integral. This finishes the proof for $\alpha = 0$ and $s' = 0$.
b) Consider the case $\alpha = 0$ and $s' = 2N \in 2\mathbb{N}\setminus\{0\}$: By the global ellipticity of P_ξ and the uniqueness in $L^2(\mathbb{R}^d, dp)$, the equation $(P_\xi - i\lambda)u = f$ has a unique solution in $\mathcal{S}'(\mathbb{R}^d)$ which belongs necessarily to $\mathcal{H}^{2N+2}$ when $f \in \mathcal{H}^{2N} \subset L^2(\mathbb{R}^d, dp)$. In $\mathcal{S}'(\mathbb{R}^d)$ and for $\varepsilon > 0$, $(P_\xi - i\lambda)\left[1+\varepsilon^2(-\Delta_p + p|^2)\right]^N u$ equals

$$\left[1+\varepsilon^2(-\Delta_p+|p|^2)\right]^N f + i\sum_{k=1}^{d}\sum_{M=1}^{N}\xi_k \begin{pmatrix} N \\ M \end{pmatrix}\left[p_k\,,\varepsilon^{2M}(-\Delta_p+|p|^2)^M\right]u\,.$$

The expression of the commutator $\left[p_k\,,\varepsilon^{2M}(-\Delta_p+|p|^2)^M\right]$ occurring in the second term as

$$\sum_{0\leq n\leq M-1,|\alpha|+|\beta|\leq 1}\varepsilon^{2(M-n)-1}c_{M,n,\alpha,\beta}(\varepsilon p)^\alpha(\varepsilon D_p)^\beta[\varepsilon^2|D_p|^2+\varepsilon^2|p|^2]^n\,,$$

is easily obtained by induction on $M \geq 1$. With $2(M-n)-1 \geq 1$ and the operator

$$(\varepsilon p)^\alpha(\varepsilon D_p)^\beta[\varepsilon^2|D_p|^2+\varepsilon^2|p|^2)]^n\left[1+\varepsilon^2(|D_p|^2+|p|^2)\right]^{-N}$$

uniformly bounded in $\mathcal{L}(L^2(\mathbb{R}^d, dp))$. The function $v = \left[1+\varepsilon^2(-\Delta_p+|p|^2)\right]^N u \in L^2(\mathbb{R}^d, dp)$ solves

$$\left[(P_\xi - i\lambda)+\sum_{k=1}^{d}\varepsilon\xi_k R_{N,k}(\varepsilon)\right]v = \tilde{f} = \left[1+\varepsilon^2(-\Delta_p+|p|^2)\right]^N f\,,$$

with $\|R_{N,k}\|_{\mathcal{L}(L^2)} \leq C_N$. Meanwhile we know $\|(P_\xi - i\lambda)^{-1}\|_{\mathcal{L}(L^2)} \leq \frac{2}{d}$. By taking $\varepsilon \leq \min\left\{\frac{1}{4C_N\langle\xi\rangle}, \sqrt{\frac{2}{d}}\right\}$, one gets

$$v = (P_\xi - i\lambda)^{-1}\left[\mathrm{Id}+\sum_{k=1}^{d}\varepsilon\xi_k R_k(P_\xi - i\lambda)^{-1}\right]^{-1}\tilde{f}\,,$$

and

$$\begin{aligned}
\varepsilon^{2N}\|u\|_{\mathcal{H}^{2N}} &\leq \left\|\left[1+\varepsilon^2(-\Delta_p+|p|^2)\right]^N u\right\| = \|v\| \\
&\leq C(1+|\xi|^{\frac{2}{3}}+|\lambda|^{\frac{1}{2}})^{-1}\left\|\left[\mathrm{Id}+\sum_{k=1}^{d}\varepsilon\xi_k R_k(P_\xi - i\lambda)^{-1}\right]^{-1}\tilde{f}\right\|, \\
&\leq 2C(1+|\xi|^{\frac{2}{3}}+|\lambda|^{\frac{1}{2}})^{-1}\|\tilde{f}\|_{L^2} \leq C'(1+|\xi|^{\frac{2}{3}}+|\lambda|^{\frac{1}{2}})^{-1}\|f\|_{\mathcal{H}^{2N}}\,.
\end{aligned}$$

Similarly we obtain

$$\begin{aligned}\varepsilon^{2N}\|u\|_{\mathcal{H}^{2N+2}} &\leq \left\|\left[1+\varepsilon^2(-\Delta_p+|p|^2)\right]^N u\right\|_{\mathcal{H}^2}\\ &\leq C''\|\tilde{f}\| \leq C^{(3)}\|f\|_{\mathcal{H}^{2N}}\,.\end{aligned}$$

We have proved

$$(1+|\xi|^{\frac{2}{3}}+|\lambda|^{\frac{1}{2}})\|(P_\xi-i\lambda)^{-1}\|_{\mathcal{L}(\mathcal{H}^{2N})}+\|(P_\xi-i\lambda)^{-1}\|_{\mathcal{L}(\mathcal{H}^{2N},\mathcal{H}^{2N+2})}\leq C'_N\langle\xi\rangle^{2N}\,.$$

c) The general result for $s'\in\mathbb{R}$ and $\alpha=0$ follows by duality and interpolation after noticing that the formal adjoint of $P_\xi-i\lambda$ is $P_{-\xi}+i\lambda$.
d) The mapping $\xi'\to(P_\xi-i\lambda)^{-1}$ is differentiable as an $\mathcal{S}$-continuous (or $\mathcal{S}'$-continuous) operator valued function of $\xi'\in\mathbb{R}^{d'}$. Its α-th derivative has the form

$$\partial_{\xi'}^\alpha(P_\xi-i\lambda)^{-1}=\sum_{\substack{(\beta_1,\ldots,\beta_{|\alpha|})\in(\mathbb{N}^d)^{|\alpha|}\,,\\ |\beta_j|=1}} c_{\beta_1,\ldots,\beta_{|\alpha|}}(P_\xi-i\lambda)^{-1}\left[\prod_{j=1}^{|\alpha|}(p')^{\beta_j}(P_\xi-i\lambda)^{-1}\right]\,.$$

From **b)** and **c)**, we deduce

$$\|(p')^{\beta_j}(P_\xi-i\lambda)^{-1}\|_{\mathcal{L}(\mathcal{H}^{s'})}\leq C_s\langle\xi\rangle^{|s'|}\,.$$

The result then follows from the estimates for $\alpha=0$ applied to the first factor $(P_\xi-i\lambda)^{-1}$. ☐

Lemma A.3. *For any $s'\in\mathbb{R}$, there exists $C_{s'}>0$ such that*

$$(1+|\xi|^{\frac{2}{3}}+|\lambda|^{\frac{1}{2}})\|(P_\xi-i\lambda)^{-1}\|_{\mathcal{L}(L^2_{s'})}+\|(-\Delta_p+|p|^2)(P_\xi-i\lambda)^{-1}\|_{\mathcal{L}(L^2_{s'})}\leq C_{s'}\,,$$

holds for all $(\xi,\lambda)\in\mathbb{R}^{d'}\times(2\pi\mathbb{Z})^{d''}\times\mathbb{R}$.

Proof. The case $s'=0$ is already proved in Lemma A.2. Assume $(P_\xi-i\lambda)u=f$ with $f\in L^2_{s'}(\mathbb{R}^d)$ and write in $\mathcal{S}'(\mathbb{R}^d)$:

$$(P_\xi-i\lambda)\langle\varepsilon p\rangle^{s'}u=\langle\varepsilon p\rangle^{s'}f-\sum_{k=1}^d\partial_{p_k}\left[\varepsilon g_{s',k}(\varepsilon p)\langle\varepsilon p\rangle^{s'}u\right]+\frac{\varepsilon^2}{2}g_{s'}(\varepsilon p)\langle\varepsilon p\rangle^{s'}u\,,$$

for $\varepsilon>0$ to be fixed, with

$$g_{s',k}(p)=\frac{\partial_{p_k}\langle p\rangle^{s'}}{\langle p\rangle^{s'}}=s'\langle p\rangle^{-2}p_k\quad\text{and}\quad g_{s'}(p)=\frac{\Delta_p\langle p\rangle^{s'}}{\langle p\rangle^{s'}}=(s'^2+(d-2)s')\langle p\rangle^{-2}.$$

We obtain

$$[(P_\xi - i\lambda) + \varepsilon R(s',\varepsilon)]\,\langle \varepsilon p\rangle^{s'} u = \langle \varepsilon p\rangle^{s'} f\,,$$

with $\|R(s',\varepsilon)\|_{\mathcal{L}(\mathcal{H}^1;L^2(\mathbb{R}^d,dp))} \leq C_{s',1}$. We know $\|(P_\xi - i\lambda)^{-1}\|_{\mathcal{L}(L^2(\mathbb{R}^d,dp);\mathcal{H}^1)} \leq C_2$ and by taking $\varepsilon_{s'} \leq \frac{1}{2C_{s',1}C_2}$ we get

$$\langle \varepsilon_{s'} p\rangle^s u = (P_\xi - i\lambda)^{-1}\left[\mathrm{Id} + \varepsilon_{s'} R(s',\varepsilon_{s'})(P_\xi - i\lambda)^{-1}\right]^{-1} \langle \varepsilon_{s'} p\rangle^s f\,.$$

This proves

$$(1 + |\xi|^{\frac{2}{3}} + |\lambda|^{\frac{1}{2}})\|\langle \varepsilon_{s'} p\rangle^{s'} u\| + \|(-\Delta_p + \frac{|p^2|}{4} + \frac{d}{2})\langle \varepsilon_{s'} p\rangle^{s'} u\| \leq C_{s'}\|\langle \varepsilon_{s'} p\rangle^{s'} f\|\,.$$

The equivalences of norms between $\|u\|_{L^2_{s'}}$ and $\|\langle \varepsilon_{s'} p\rangle^{s'} u\|$, and between $\|(-\Delta_p + |p|^2)\langle \varepsilon_{s'} p\rangle^{s'} u\|$ and $\|(\frac{-\Delta_p + |p|^2 + d}{2})u\|_{L^2_{s'}} = \|\langle p\rangle^{s'}(\frac{-\Delta_p + |p|^2 + d}{2})u\|$ finally yields the result. $\square$

Theorem A.1 can be used to solve $(P_\pm - i\lambda)u = f$ in the distributional sense. In particular when $Q = \mathbb{R}\times Q'$ with $Q' = \mathbb{R}^{d_1-1}\times\mathbb{T}^{d_2}$, $d = d_1 + d_2$, with the coordinates $q = (q^1, q', p_1, p')$ on $X = T^*Q$, and $\gamma \in L^2(Q'\times\mathbb{R}^d, \frac{dq'dp}{|p_1|}) \subset L^2(Q'\times\mathbb{R}^d, \frac{dq'dp}{\langle p\rangle})$ the distribution $\gamma(q',p)\delta_0(q_1)$ belongs to $H^{s_0}(Q; L^2_{-\frac{1}{2}})$ for any $s_0 < -\frac{1}{2}$. The equation $(P_\pm - i\lambda)u = \gamma(q',p)\delta_0(q_1)$ then admits a unique solution in $H^{s_0}(Q; L^2_{-\frac{1}{2}})$ with

$$\begin{aligned}\||D_p|^2 u\|_{H^{s_0}(Q;L^2_{-\frac{1}{2}})} + \||p|^2 u\|_{H^{s_0}(Q;L^2_{-\frac{1}{2}})} + \langle\lambda\rangle^{\frac{1}{2}}\|u\|_{H^{s_0}(Q;L^2_{-\frac{1}{2}})}&\\ + \|u\|_{H^{s_0+\frac{2}{3}}(Q;L^2_{-\frac{1}{2}})} \leq C_{s_0}\|\gamma\|_{L^2(Q'\times\mathbb{R}^d,\frac{dq'dp}{|p_1|})}\,.&\qquad(127)\end{aligned}$$

This estimate can be given a more precise form with a simple integration by parts.

Proposition A.4. *Assume $Q = \mathbb{R}\times Q'$ with $Q' = \mathbb{R}^{d_1-1}\times\mathbb{T}^{d_2}$, $d_1 \geq 1$, $d = d_1 + d_2$, with the coordinates $q = (q^1, q', p_1, p')$ on $X = T^*Q$.*
For any $u \in D(K_\pm)$, the following trace estimate holds

$$\langle\lambda\rangle^{\frac{1}{4}}\|u(q_1 = 0, q', p)\|_{L^2(Q'\times\mathbb{R}^d,|p_1|dq'dp)} \leq C'\|(K_\pm - i\lambda)u\|\,. \qquad (128)$$

The dual version says that for any $\gamma \in L^2(Q'\times\mathbb{R}^d, \frac{dq'dp}{|p_1|})$, the solution u to $(P_\pm - i\lambda)u = \gamma(q',p)\delta_0(q_1)$ belongs to $L^2(X, dqdp)$ with the estimate

$$\langle\lambda\rangle^{\frac{1}{4}}\|u\| \leq C''\|\gamma\|_{L^2(Q'\times\mathbb{R}^d,\frac{dq'dp}{|p_1|})}\,. \qquad (129)$$

Proof. Set $X_- = (-\infty, 0) \times Q' \times \mathbb{R}^d$. For $u \in \mathcal{C}_0^\infty(X)$ and $\lambda \in \mathbb{R}$, write

$$\begin{aligned}\int_{Q'\times\mathbb{R}^d} |p_1||u(0,q',p)|^2\, dq'dp \quad &= 2\,\mathbb{R}\mathrm{e} \int_{X_-} |p_1|\partial_{q_1}\overline{u}(q,p)u(q,p)\, dqdp \\ &= \pm 2\,\mathbb{R}\mathrm{e}\int_X (\pm p.\partial_q + i\lambda)\overline{u}(q,p)\,\mathrm{sign}\,(p_1)u(q,p)\, dqdp \\ &= \pm 2\,\mathbb{R}\mathrm{e}\,\langle(\pm p\partial_q - i\lambda)u\,,\, 1_{\mathbb{R}_-}(q)\,\mathrm{sign}\,(p)u\rangle \\ &\leq 2\|(\pm p\partial_q - i\lambda)u\|\|u\| \leq C\langle\lambda\rangle^{-\frac{1}{2}}\|(K_\pm - i\lambda)u\|^2\,,\end{aligned}$$

by applying (123) with $s, s' = 0$. The extension of the estimate to any $u \in D(K_\pm)$ is a consequence of the density of $\mathcal{C}_0^\infty(X)$ in $D(K_\pm)$.
The existence and uniqueness of a solution $u \in L^2(X, dqdp)$ to $(P_\pm - i\lambda)u = \gamma(q',p)\delta_0(q_1)$ comes from Theorem A.1 as explained for (127). For any $\phi \in L^2(Q' \times \mathbb{R}^d, |p_1|dq'dp)$ and any $f \in L^2(X, dqdp)$ the previous result applied to $K_\pm^* = K_\mp$ gives

$$\begin{aligned}&\left|\int_{Q'\times\mathbb{R}^d} \overline{\phi(q',p)}[(K_\pm^* + i\lambda)^{-1}f](0,q',p)|p_1|dq'dp\right| \\ &\qquad\qquad\qquad \leq C'\langle\lambda\rangle^{-\frac{1}{4}}\|\phi\|_{L^2(Q'\times\mathbb{R}^d, |p_1|dq'dp)}\|f\|\,.\end{aligned}$$

But the above integral equals

$$\begin{aligned}&\int_{\mathbb{R}} \overline{\phi(q',p)}[(K^* + i\lambda)^{-1}f](0,q',p)|p_1|dq'dp \\ &\qquad = \langle\phi(q',p)|p_1|\delta_0(q_1)\,,\,(K_\pm^* + i\lambda)^{-1}f\rangle \\ &\qquad\qquad = \langle(P_\pm - i\lambda)^{-1}[\phi(q',p)|p_1|\delta_0(q_1)]\,,\, f\rangle\,.\end{aligned}$$

After setting $\gamma = |p_1|\phi(q',p)$, we obtain

$$\begin{aligned}\left|\langle f\,,\,(P_\pm - i\lambda)^{-1}[\gamma(q',p)\delta_0(q_1)]\rangle\right| &\leq C''\langle\lambda\rangle^{-\frac{1}{4}}\|f\|\||p_1|^{-1}\gamma\|_{L^2(Q'\times\mathbb{R}^d, |p_1|dq'dp)} \\ &= C''\langle\lambda\rangle^{-\frac{1}{4}}\|f\|\|\gamma\|_{L^2(Q'\times\mathbb{R}^d, \frac{dq'dp}{|p_1|})}\,,\end{aligned}$$

which ends the proof. ∎

We end with another application of (127).

Proposition A.5. *Assume* $Q = \mathbb{R} \times Q'$ *with* $Q' = \mathbb{T}^{d'} \times \mathbb{R}^{d''-1}$, $d'' \geq 1$,$d = d' + d''$, *and take the coordinates* $(q^1, q', p_1, p') \in \mathbb{R} \times Q' \times \mathbb{R} \times \mathbb{R}^{d-1}$ *in*

$X = T^*Q$.
If $u \in L^2(Q;\mathcal{H}^1)$ solves

$$(P_\pm - i\lambda)u = \gamma(q', p)\delta_0(q^1)$$

with $\gamma \in L^2(Q' \times \mathbb{R}^d, \frac{dq'dp}{|p_1|})$. Then for any $s \in [0, \frac{1}{9})$, u belongs to $H^s(Q;\mathcal{H}^0)$ with the estimate

$$\|u\|_{H^s(Q;\mathcal{H}^0)} \le C_s \left[\|\gamma\|_{L^2(Q'\times\mathbb{R}^d, \frac{dq'dp}{|p_1|})} + \|u\|_{L^2(Q;\mathcal{H}^1)} .\right]$$

When $d'' = 1$, $R \in (0, +\infty)$ and $s \in (0, \frac{1}{9})$, the embedding $H^s(Q;\mathcal{H}^0) \cap L^2(Q;\mathcal{H}^1) \subset L^2([-R,R] \times Q' \times \mathbb{R}^d; dqdp)$ is compact.

Proof. From (127) we deduce

$$\|\langle p\rangle^{-\frac{1}{2}}u\|_{H^\nu(Q;\mathcal{H}^0)} \le \|u\|_{H^\nu(Q;L^2_{-\frac{1}{2}})} \le C_\nu \|\gamma\|_{L^2(Q'\times\mathbb{R}^d, \frac{dq'dp}{|p_1|})} ,$$

for all $\nu \in [0, \frac{1}{6})$. The assumption $u \in L^2(Q;\mathcal{H}^1)$ implies

$$\|\langle p\rangle u\|_{L^2(X,dqdp)} \le C_d \|u\|_{L^2(Q;\mathcal{H}^1)} .$$

With

$$\|u\|_{H^{\frac{2}{3}\nu}(Q;\mathcal{H}^0)} \le C'_\nu \|\langle p\rangle^{-\frac{1}{2}}u\|^{\frac{2}{3}}_{H^\nu(Q;\mathcal{H}^0)} \|\langle p\rangle u\|^{\frac{1}{3}}_{L^2(X,dqdp)}$$

we choose $s = \frac{2}{3}\nu \subset [0, \frac{1}{9})$. We obtain

$$\begin{aligned} \|u\|_{H^s(Q;\mathcal{H}^0)} &\le C_s \|\gamma\|^{\frac{2}{3}}_{L^2(Q'\times\mathbb{R}^d, \frac{dq'dp}{|p_1|})} \|u\|^{\frac{1}{3}}_{L^2(Q;\mathcal{H}^1)} \\ &\le C'_s \left[\|\gamma\|_{L^2(Q'\times\mathbb{R}^d, \frac{dq'dp}{|p_1|})} + \|u\|_{L^2(Q;\mathcal{H}^1)}\right] . \end{aligned}$$

With $[-R,R] \times Q' = [-R,R] \times \mathbb{T}^{d-1}$, the compactness statement is obvious. $\square$

B Partitions of unity

We briefly review a few formulas with partitions of unities. We shall work on a riemannian manifold M possibly with boundary and $L^2(M)$ is endowed with the scalar product $\langle u\,,\, v\rangle = \int_M \overline{u}v\; d\mathrm{Vol}_g$ and the norm $\|u\| = \sqrt{\langle u\,,\, u\rangle}$, associated with the metric g. The operator $P=\sum_{|\alpha|\leq m}a_\alpha(x)\partial_x^\alpha$ is a differential operator with $\mathcal{C}^\infty$-coefficients and its formal adjoint is denoted by P^*. The family $(\chi_\ell)_{\ell\in\mathcal{L}}$ is a locally finite family $\mathcal{C}_0^\infty(M;\mathbb{R})$ such that $\sum_{\ell\in\mathcal{L}}\chi_\ell^2\equiv 1$. For every $\ell\in\mathcal{L}$, P_ℓ will denote a local model of P, that is $P\chi_\ell=P_\ell\chi_\ell$. We also assume that possible boundary conditions behave well with the partition of unity. The following formula are written for $u\in\mathcal{C}_0^\infty(M)$ (or possibly $u\in\mathcal{C}_0^\infty(\overline{M})$ fulfilling some boundary conditions) or $u\in L^2(M)$ provided that all the terms make sense in $\mathcal{D}'(M)$ or in $L^2(M)$. Finally the notation $\mathrm{ad}_A^k B$ is defined by $\mathrm{ad}_A B=[A,B]$ and $\mathrm{ad}_A^k B=\mathrm{ad}_A\mathrm{ad}_A^{k-1}B$.

$$\|Pu\|^2-\sum_{\ell\in\mathcal{L}}\|P_\ell\chi_\ell u\|^2=-\sum_{\ell\in\mathcal{L}}\|(\mathrm{ad}_{\chi_\ell}P)u\|^2+\mathrm{Re}\,\langle Pu\,,(\mathrm{ad}_{\chi_\ell}^2P)u\rangle\,, \tag{130}$$

$$P-\sum_{\ell\in\mathcal{L}}\chi_\ell P_\ell\chi_\ell=-\frac{1}{2}\sum_{\ell\in\mathcal{L}}\mathrm{ad}_{\chi_\ell}^2P\,. \tag{131}$$

For (131) write

$$P-\sum_{\ell\in\mathcal{L}}\chi_\ell P\chi_\ell=\sum_{\ell\in\mathcal{L}}[P,\chi_\ell]\,\chi_\ell=\sum_{\ell\in\mathcal{L}}\chi_\ell\,[\chi_\ell,P]$$

and take half the sum of the two last expressions with $[P,\chi_\ell]\,\chi_\ell+\chi_\ell[\chi_\ell,P]=-\mathrm{ad}_{\chi_\ell}^2P$.
For (130) introduce the formal adjoints P^* of P (resp P_ℓ^* of P_ℓ) and write

$$\|Pu\|^2-\sum_{\ell\in\mathcal{L}}\|P_\ell\chi_\ell u\|^2=\sum_{\ell\in\mathcal{L}}\langle u\,,\,[P^*\chi_\ell^2P-\chi_\ell P^*P\chi_\ell]u\rangle\,.$$

For one given $\ell\in\mathcal{L}$ compute

$$\begin{aligned}P^*\chi_\ell^2P-\chi_\ell P^*P\chi_\ell &= P^*\chi_\ell\,[\chi_\ell,P]+[P^*,\chi_\ell]\,P\chi_\ell\\ &= P^*\chi_\ell\,[\chi_\ell,P]+[P^*,\chi_\ell]\,\chi_\ell P+[P^*,\chi_\ell]\,[P,\chi_\ell]\,.\end{aligned}$$

For the second and third term, use

$$[P^*,\chi_\ell]=[\chi_\ell,P]^*=-[P,\chi_\ell]^*\,.$$

For the first term, use

$$2\chi_\ell[\chi_\ell,P]=\left[\chi_\ell^2,P\right]-[[\chi_\ell,P]\,,\chi_\ell]=\left[\chi_\ell^2,P\right]+\mathrm{ad}_{\chi_\ell}^2P\,.$$

This gives

$$\|\chi_\ell P\|^2 - \|P\chi_\ell u\|^2 = \mathbb{R}\mathrm{e}\,\langle Pu\,,\,\left[\chi_\ell^2, P\right]u\rangle + \mathbb{R}\mathrm{e}\,\langle Pu\,,\,(\mathrm{ad}\,^2_{\chi_\ell}P)u\rangle - \|(\mathrm{ad}\,_{\chi_\ell}P)u\|^2_{L^2}\,,$$

and summing over $\ell \in \mathcal{L}$ proves (130). We conclude with a duality and interpolation argument.

Lemma B.1. *Let $E_1 \subset E_0 \subset E_{-1}$ and $F_1 \subset F_0 \subset F_{-1}$ be two Hilbert triples (the inclusion are continuous and dense embeddings, the $_{-1}$ space being the dual of the $_{+1}$ space). The interpolated Hilbert spaces are denoted by E_r, F_r for any $r \in [-1, 1]$. Let $J : E_0 \to F_0$ be an isometry, $J^{*0}J = Id_{E_0}$ (non necessarily surjective), such that*

$$\exists C > 0\,, \forall u \in E_1\,, \quad C_1^{-1}\|u\|_{E_1} \leq \|Ju\|_{F_1} \leq C_1\|u\|_{E_1}\,.$$

Then for any $r \in [-1, 1]$, J defines a bounded operator from E_r to F_r and the equivalence

$$\forall u \in E_r\,, \quad C_r^{-1}\|u\|_{E_r} \leq \|Ju\|_{F_r} \leq C_r\|u\|_{E_r}\,,$$

holds with $C_r = C_1^{|r|}$.

Proof. The equivalence of $\|Ju\|_{F_1}$ and $\|u\|_{E_1}$ implies that $J\big|_{E_1}$ has a closed range G_1 in F_1 with the inverse $J^{*0}\big|_{G_1}$. Let Π_{G_1} a continuous projection and note

$$|\langle Jv\,,\,u\rangle| = |\langle v\,,\,J^{*0}(\Pi_{G_1}u)\rangle| \leq C\|v\|_{E_{-1}}\|u\|_{F_1}\,.$$

Therefore J extends as a bounded operator from $E_{-1} \to F_{-1}$. By interpolation $J \in \mathcal{L}(E_r, F_r)$ and $J^{*0} \in \mathcal{L}(F_r, E_r)$ with $J^{*0}J = \mathrm{Id}_{E_r}$. ☐

Application: If $(\chi_\ell)_{\ell\in\mathcal{L}}$ is a locally finite partition of unity such that $\sum_{\ell\in\mathcal{L}}\chi_\ell^2 = 1$ and $(A, D(A))$ is positive self-adjoint operator in $L^2(M)$ such that

$$\left(\frac{\sum_{\ell\in\mathcal{L}}\|A\chi_\ell u\|^2}{\|Au\|^2}\right)^{\pm 1} \leq C\,,$$

then the equivalence

$$\left(\frac{\sum_{\ell\in\mathcal{L}}\|A^r\chi_\ell u\|^2}{\|A^r u\|^2}\right)^{\pm 1} \leq C_r$$

holds for any $r \in [-1,1]$. Simply apply the previous lemma with

$$
\begin{aligned}
& E_0 = L^2(M) \quad , \quad E_1 = D(A) \quad , \quad E_{-1} = D(A)' \,, \\
& F_{0,\pm 1} = \bigoplus_{\ell \in \mathcal{L}} E_{0,\pm 1} \,, \\
\text{and} \qquad & Ju = (\chi_\ell u)_{\ell \in \mathcal{L}} \,.
\end{aligned}
$$

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